Journal of Combinatorial Theory, Series A 132 (2015) 188–193

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Clique vectors of k-connected chordal graphs

Afshin Goodarzi

Royal Institute of Technology, Department of Mathematics, S-100 44, Stockholm, Sweden

ARTICLE INFO

Article history:
Received 3 June 2014
Available online 22 January 2015

Dedicated to Ralf Fröberg on the occasion of his 70th birthday

Keywords:
Clique vectors
Chordal graphs
Graph connectivity

ABSTRACT

The clique vector $\mathfrak{c}(G)$ of a graph G is the sequence $(c_1, c_2, \dots, c_d)$ in $\mathbb{N}^d$, where c_i is the number of cliques in G with i vertices and d is the largest cardinality of a clique in G. In this note, we use tools from commutative algebra to characterize all possible clique vectors of k-connected chordal graphs.

1. Introduction

The clique vector of a graph G is an interesting numerical invariant assigned to G. The study of clique vectors goes back at least to Zykov's generalization of Turán's graph theorem [8]. The clique vector of G is by definition the f-vector of its clique complex. Challenging problems including the Kalai–Eckhoff conjecture and the classification of the f-vector of flag complexes led many researchers to investigate clique vectors, see for instance [2,3,6]. While the Kalai–Eckhoff conjecture is now settled by Frohmader [2], the latter problem is still wide open.

Herzog et al. [6] characterized all possible clique vectors of chordal graphs. A graph G is called *k-connected* if it has at least k vertices and removing any set of vertices of G of cardinality less than k yields a connected graph. Thus a 1-connected graph is simply

E-mail address: afshingo@kth.se.

http://dx.doi.org/10.1016/j.jcta.2015.01.001

a connected graph. We use the convention that every graph is 0-connected. The *connectivity number* $\kappa(G)$ of G is the maximum number k such that G is k-connected. The aim of this paper is to characterize all possible clique vectors of k-connected chordal graphs. More precisely we prove the following result.

Theorem 1. *A vector $\mathfrak{c} = (c_1, \ldots, c_d) \in \mathbb{N}^d$ is the clique vector of a k-connected chordal graph if and only if the vector $\mathfrak{b} = (b_1, \ldots, b_d)$ defined by*

$$\sum_{i=1}^{d} b_i (x+1)^{i-1} = \sum_{i=1}^{d} c_i x^{i-1} \tag{1}$$

has positive components and $b_1 = b_2 = \cdots = b_k = 1$.

The theorem above is a refinement of [6, Theorem 1.1], in the sense that putting $k = 0$, the only requirement on b-numbers is to be positive, so [6, Theorem 1.1] will be obtained.

In order to prove our main result, we shall use techniques from algebraic shifting theory to reduce the problem to the class of shifted graphs, the so called threshold graphs.

The rest of this paper is organized as follows. In Section 2, we verify the characterization for threshold graphs by giving a combinatorial interpretation of the b-numbers. Section 3 is devoted to a study of the connectivity of a graph via certain homological invariants of a ring associated to it. Finally, in Section 4 we prove our main result.

All undefined algebraic terminology can be found in the book of Herzog and Hibi [5].

2. Clique vectors of threshold graphs

Let G be a graph. We denote by $S(G)$ the graph obtained from G by adding a new vertex and connecting it to all vertices of G. Also, we denote by $D(G)$ the graph obtained from G by adding an isolated vertex. Clearly the numbers of i-cliques in G and $D(G)$ are the same, unless $i = 1$. On the other hand, it is easy to verify the following formula that relates the numbers of cliques in G and $S(G)$:

$$1 + \sum_i c_i\big(S(G)\big)x^i = \left(1 + \sum_i c_i(G)x^i\right)(1 + x). \tag{2}$$

A graph T is called *threshold*, if it can be obtained from the null graph by a sequence of S- and D-operators. Thus, we have a bijection between threshold graphs and words on the alphabet $\{S, D\}$ (reading from left to right) with an S in its final (rightmost) position.[1] Clearly, every threshold graph is chordal.

[1] The D- and S-operations on the null graph, i.e. the graph having zero vertices, result the same graph. So, to have a unique representation of each threshold graph, we may assume that the operation D is allowed when the graph is not null.