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Intersecting families of discrete structures are typically trivial

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ABSTRACT

The study of intersecting structures is central to extremal combinatorics. A family of permutations $\mathcal{F} \subset S_n$ is t-intersecting if any two permutations in $\mathcal{F}$ agree on some t indices, and is trivial if all permutations in $\mathcal{F}$ agree on the same t indices. A k-uniform hypergraph is t-intersecting if any two of its edges have t vertices in common, and trivial if all its edges share the same t vertices.

The fundamental problem is to determine how large an intersecting family can be. Ellis, Friedgut and Pilpel proved that for n sufficiently large with respect to t, the largest t-intersecting families in S_n are the trivial ones. The classic Erdős–Ko–Rado theorem shows that the largest t-intersecting k-uniform hypergraphs are also trivial when n is large. We determine the typical structure of t-intersecting families, extending these results to show that almost all intersecting families are trivial. We also obtain sparse analogues of these extremal results, showing that they hold in random settings.

Our proofs use the Bollobás set-pairs inequality to bound the number of maximal intersecting families, which can then

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be combined with known stability theorems. We also obtain similar results for vector spaces.

1. Introduction

The fundamental problem in extremal combinatorics asks how large a system can be under certain restrictions. Once resolved, this can then be strengthened by enumerating such systems and describing their typical structure. In the context of graph theory, this study was initiated by Erdős, Kleitman and Rothschild [12] in 1976, who proved that almost all triangle-free graphs are bipartite. In extremal set theory, a landmark result was the determination of the number of antichains among subsets of an n-element set by Kleitman [22] in 1969. These results have since inspired a great deal of research over the years, with many classic theorems having been so extended.

Intersecting hypergraphs were first studied in the seminal 1961 paper of Erdős, Ko and Rado [13]. Not only have versions of the Erdős–Ko–Rado theorem been obtained in various other settings, including permutations and vector spaces, but a great deal of modern research is still devoted to proving further extensions. In this paper, we study intersecting families of discrete systems in various settings, determining their typical structure as n, the size of the underlying ground set, tends to infinity.

We will now present our results and briefly review the extremal results regarding intersecting families in these different settings. We discuss permutations in Section 1.1, hypergraphs in Section 1.2 and vector spaces in Section 1.3.

In what follows, we write log for logarithms to the base 2, and ln for logarithms to the base e.

1.1. Permutations

Denote by S_n the symmetric group on $[n]$. A family of permutations $\mathcal{F} \subseteq S_n$ is *t-intersecting* if any two permutations in $\mathcal{F}$ agree on at least t indices; that is, for any $\sigma, \pi \in \mathcal{F}$, $|\sigma \cap \pi| = |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. When $t = 1$, we simply call such families *intersecting*. A natural example of a t-intersecting family $\mathcal{F} \subseteq S_n$ is a *trivial* t-intersecting family, where there is a fixed t-set $I \subseteq [n]$ and values $\{x_i : i \in I\}$ such that for every $\sigma \in \mathcal{F}$ and $i \in I$, $\sigma(i) = x_i$. Ellis, Friedgut and Pilpel [11] proved that, for n sufficiently large with respect to t, a t-intersecting family $\mathcal{F} \subseteq S_n$ has size at most $(n-t)!$, with equality only if $\mathcal{F}$ is trivial. Our first result determines the typical structure of t-intersecting families in S_n, showing that trivial families are not just extremal but also typical.