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Spanning forests in regular planar maps ☆

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A B S T R A C T

We address the enumeration of p-valent planar maps equipped with a spanning forest, with a weight z per face and a weight u per connected component of the forest. Equivalently, we count p-valent maps equipped with a spanning *tree*, with a weight z per face and a weight $\mu := u+1$ per *internally active* edge, in the sense of Tutte; or the (dual) p-angulations equipped with a *recurrent sandpile configuration*, with a weight z per vertex and a variable $\mu := u + 1$ that keeps track of the *level* of the configuration. This enumeration problem also corresponds to the limit $q \to 0$ of the q-state Potts model on p-angulations.

Our approach is purely combinatorial. The associated generating function, denoted $F(z, u)$, is expressed in terms of a pair of series defined implicitly by a system involving doubly hypergeometric series. We derive from this system that $F(z, u)$ is *differentially algebraic* in z, that is, satisfies a differential equation in z with polynomial coefficients in z and u. This has recently been proved to hold for the more general Potts model on 3-valent maps, but via a much more involved and less combinatorial proof.

For $u \geq -1$, we study the singularities of $F(z, u)$ and the corresponding asymptotic behaviour of its nth coefficient. For $u > 0$, we find the standard behaviour of planar maps, with a subexponential term in $n^{-5/2}$. At $u = 0$ we witness a phase transition with a term n^{-3}. When $u \in [-1, 0)$, we obtain an extremely unusual behaviour in $n^{-3}(\ln n)^{-2}$. To our knowledge, this is a new "universality class" for planar maps. We

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analyze the phase transition at $u = 0$ in terms of the sandpile model on large maps, and find it to be of infinite order.

1. Introduction

A *planar map* is a proper embedding of a connected graph in the sphere. The enumeration of planar maps has received a continuous attention in the past 60 years, first in combinatorics with the pioneering work of Tutte [54], then in theoretical physics [24], where maps are considered as random surfaces modelling the effect of *quantum gravity*, and more recently in probability theory [42,44]. General planar maps have been studied, as well as sub-families obtained by imposing constraints of higher connectivity, or prescribing the degrees of vertices or faces (*e.g.*, triangulations). Precise definitions are given below.

Several robust enumeration methods have been designed, from Tutte's recursive approach (*e.g.* [53]), which leads to functional equations for the generating functions of maps, to the beautiful bijections initiated by Schaeffer [49], and further developed by physicists and combinatorics alike [11,21], via a powerful approach based on matrix integrals [32]. See for instance [18] for a more complete (though non-exhaustive) bibliography.

Beyond the enumerative and asymptotic properties of planar maps, which are now well understood, the attention has also focused on two more general questions: maps on higher genus surfaces [6,26], and maps equipped with an additional structure. The latter question is particularly relevant in physics, where a surface on which nothing happens ("pure gravity") is of little interest. For instance, one has studied maps equipped with a polymer [34], with an Ising model [21,40,17,20] or more generally a Potts model, with a proper colouring [55,56], with loop models [16,15], with a spanning tree [47], or percolation on planar maps [2,10]. These articles parallel the analogous investigation, started in the 80s, of the same list of models on regular planar lattices (square, triangular...), based at the time on the techniques of conformal field theory (CFT).

In particular, several papers have been devoted in the past 20 years to the study of the Potts model on families of planar maps [4,14,31,35,39,58]. One motivation, mentioned by one of our referees, is that "the q-state Potts model (for $0 \leq q \leq 4$) covers the full range of universality classes interpolating the discrete family of $(p, p+1)$ minimal CFT's". In combinatorial terms, solving this model means counting maps equipped with a vertex-colouring in q colours, according to the size (*e.g.*, the number of edges) and the number of *monochromatic edges* (edges whose endpoints have the same colour). Up to a change of variables, this also means counting maps weighted by their *Tutte polynomial*, a bivariate combinatorial invariant which has numerous interesting specializations. By generalizing Tutte's formidable solution of properly coloured triangulations (1973–1982), it has recently been proved that the Potts generating function is *differentially algebraic*,