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The behavior of Stanley depth under polarization

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Let $\mathbb{K}$ be a field, $R = \mathbb{K}[X_1, \dots, X_n]$ be the polynomial ring and $J \subsetneq I$ be two monomial ideals in R. In this paper we show that

$$\operatorname{sdepth} \ I/J - \operatorname{depth} \ I/J = \operatorname{sdepth} \ I^p/J^p - \operatorname{depth} \ I^p/J^p,$$

where $\operatorname{sdepth} I/J$ denotes the Stanley depth and I^p denotes the polarization. This solves a conjecture by Herzog [9] and reduces the famous Stanley conjecture (for modules of the form I/J) to the squarefree case. As a consequence, the Stanley conjecture for algebras of the form R/I and the well-known combinatorial conjecture that every Cohen–Macaulay simplicial complex is partitionable are equivalent.

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1. Introduction

In 1982, R. Stanley conjectured in his celebrated paper [26] an upper bound for the depth of a multigraded module of combinatorial nature, called *Stanley depth* later on. A proof of this conjecture turned out to be a difficult problem: it began soon to be called the *Stanley conjecture*. Since then, several authors began to study intensively this problem, starting with the reformulation by Apel of the most important cases of the conjecture, i.e. the Stanley conjecture for a monomial ideal I and for the factor ring R/I, see [2, Conjecture 2] and [3, Conjecture 1]. Afterwards, most of the research concentrates on the particular case of a module of the form I/J for two monomial ideals $J \subsetneq I$ in the polynomial ring $R = \mathbb{K}[X_1, \dots, X_n]$ over some field $\mathbb{K}$; motivated by works of Herzog and Popescu [11,23], the Stanley conjecture became one important open problem in algebra and combinatorics.

A natural first step to approach the Stanley conjecture is to try to reduce it to squarefree monomial ideals. The arguable most straightforward method for this is via polarization. This is a process which replaces an arbitrary monomial ideal I with a certain squarefree monomial ideal I^p, such that I can be recovered from I^p by dividing out a regular sequence. The behavior of many invariants of I under polarization is well understood. In particular, as polarization preserves the projective dimension, the change in the depth is just the change in the number of variables. In view of the Stanley conjecture, one would hope for a similar behavior of the Stanley depth. This was formulated as a conjecture by Herzog in the survey [9] as follows:

Conjecture 1.1. *(See Conjecture 62, [9].) Let $I \subset R$ be a monomial ideal. Then*

$$\operatorname{sdepth} R/I - \operatorname{depth} R/I = \operatorname{sdepth} R^p/I^p - \operatorname{depth} R^p/I^p$$

where R^p is the ring where I^p is defined.

However, despite the naturality of the question and a considerable effort, it remained open for quite some time. The main result of our paper (Theorem 4.4) is the proof of Conjecture 1.1. We show it even more generally for modules of the form I/J for two monomial ideals $J \subsetneq I \subseteq R$.

This has two important consequences. First, it immediately follows that I/J satisfies the Stanley conjecture if and only if its polarization I^p/J^p does so, cf. Corollary 4.5. Thus the Stanley conjecture for modules of the form I/J—in particular [2, Conjecture 2] and [3, Conjecture 1]—is effectively reduced to the squarefree case.

For the second consequence, as noted by Stanley himself in [26, p. 191], the Stanley conjecture was formulated such that "the question raised in [25, p. 149, line 6] or [8, Rmk. 5.2] would follow affirmatively". This question was reformulated by Stanley [27, Conjecture 2.7], and asks whether every Cohen–Macaulay simplicial complex is partitionable. While it is clear that the Stanley conjecture implies the Garsia–Stanley conjecture