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A probabilistic threshold for monochromatic arithmetic progressions

Aaron Robertson

Department of Mathematics, Colgate University, Hamilton, NY, United States

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ABSTRACT

Let $f_r(k) = \sqrt{k} \cdot r^{k/2}$ (where $r \geq 2$ is fixed) and consider r-colorings of $[1, n_k] = \{1, 2, \dots, n_k\}$. We show that $f_r(k)$ is a threshold function for k-term arithmetic progressions in the following sense: if $n_k = \omega(f_r(k))$, then $\lim_{k\to\infty} P([1, n_k]$ contains a monochromatic k-term arithmetic progression$) = 1$; while, if $n_k = o(f_r(k))$, then $\lim_{k\to\infty} P([1, n_k]$ contains a k-term monochromatic arithmetic progression$) = 0$.

1. Introduction

For $k, r \in \mathbb{Z}^+$, let $w(k; r)$ be the minimum integer such that *every* r-coloring of $[1, w(k; r)]$ admits a monochromatic k-term arithmetic progression. The existence of such an integer was shown by van der Waerden [10], and these integers are referred to as van der Waerden numbers. Current knowledge (for r fixed) places $w(k; r)$ somewhere between $\frac{r^{k-1}}{ek}(1 + o(1))$ and

$$2^{2^{r^{2^{2^{k+9}}}}},$$

E-mail address: arobertson@colgate.edu.

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with the upper bound being from one of Gowers' seminal works [5] (with slightly better lower bounds when $r = 2$). A matching of upper and lower bounds appears unlikely in the near (or distant?) future. However, by loosening the restriction that *every* r-coloring must have a certain property to *almost every* (in the probabilistic sense), we are able to home in on the rate of growth of the associated numbers.

Let $\mathcal{C}_n$ be the collection of all r-colorings of $[1, n]$ and define $\mathcal{C} = \bigcup_{i \geq 1} C_i$. A *threshold function* on $\mathcal{C}$ for a property $\mathcal{P}$ is a function $t(n)$ such that the following both hold:

- given $f(n) = \omega(t(n))$ and $S \in \mathcal{C}$ with $|S| = f(n)$ we have $\lim_{n\to\infty} P(S$ has property $\mathcal{P}) = 1$;
- given $g(n) = o(t(n))$ and $S \in \mathcal{C}$ with $|S| = g(n)$ we have $\lim_{n\to\infty} P(S$ has property $\mathcal{P}) = 0$.

Although the majority of threshold functions studied have concerned graphs (which would require a change in the definition of $\mathcal{C}$ above), work on integer Ramsey structures has been investigated. Conlon and Gowers [3] proved a probability threshold function in the area of random subsets containing a k-term arithmetic progression. Schacht [9] has a similar result, derived independently using different methods. Balogh, Morris, and Samotij [1] provide yet another method addressing the same problem, as do Saxton and Thomason [8]. Friedgut, Rödl, and Schacht [4] provide probability thresholds for Rado's Theorem.

From a different perspective, a threshold function for the size of a random subset needed to admit a monochromatic arithmetic progressions is given by Rödl and Ruciński [6,7]. The present article is more in-line with this direction insomuch as we are coloring integers at random rather than searching for the probability threshold with which we include (i.e., "color") a particular integer.

In this article, we assume that every r-coloring of a given interval is equally likely, i.e., for each integer in the interval, the probability that it is a given color is $\frac{1}{r}$. We refer to a k-term arithmetic progression as a k-ap and will use the notation $\langle a, d \rangle_k$ to represent $a, a + d, a + 2d, \ldots, a + (k-1)d$, where we refer to d as the *gap* of the k-ap. We use the standard notation $[1, n] = \{1, 2, \ldots, n\}$.

For ease of exposition, we separate the definition of a threshold function into parts via the following two definitions.

Definition 1. Let $t(k)$ be a function defined on $\mathbb{Z}^+$ with some property $\mathcal{P}$. We say that $t(k)$ is a *minimal function* (with respect to $\mathcal{P}$) if for every function $s(k)$ defined on $\mathbb{Z}^+$ with property $\mathcal{P}$ we have

$$\liminf_{k\to\infty} \frac{t(k)}{s(k)} \leq 1.$$

We say that $t(k)$ is a *maximal function* (with respect to $\mathcal{P}$) if for every function $s(k)$ defined on $\mathbb{Z}^+$ with property $\mathcal{P}$ we have