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On some variations of coloring problems of infinite words

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ABSTRACT

Given a finite coloring (or finite partition) of the free semigroup $\mathbb{A}^+$ over a set $\mathbb{A}$, we consider various types of monochromatic factorizations of right sided infinite words $x \in \mathbb{A}^\omega$. Some stronger versions of the usual notion of monochromatic factorization are introduced. A factorization is called sequentially monochromatic when concatenations of consecutive blocks are monochromatic. A sequentially monochromatic factorization is called ultra monochromatic if any concatenation of arbitrary permuted blocks of the factorization has the same color of the single blocks. We establish links, and in some cases equivalences, between the existence of these factorizations and fundamental results in Ramsey theory including the infinite Ramsey theorem, Hindman's finite sums theorem, partition regularity of IP sets and the Milliken–Taylor theorem. We prove that for each finite set $\mathbb{A}$ and each finite coloring $\varphi : \mathbb{A}^+ \to C$, for almost all words $x \in \mathbb{A}^\omega$, there exists y in the subshift generated by x admitting a φ-ultra monochromatic factorization, where "almost all" refers to the Bernoulli measure on $\mathbb{A}^\omega$.

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1. Introduction and preliminaries

Let $\mathbb{A}$ be a non-empty set, or *alphabet*, and $\mathbb{A}^+$ denote the *free semigroup* over $\mathbb{A}$, i.e., the set of all finite words $x = x_0x_1 \cdots x_n$ with $x_i \in \mathbb{A}$, $0 \le i \le n$. Adding to $\mathbb{A}^+$ an identity element ε, usually called *empty word*, one obtains the free monoid $\mathbb{A}^*$. Let $\mathbb{A}^\omega$ denote the set of all right sided infinite words $x = x_0x_1 \cdots$ with $x_i \in \mathbb{A}$, $i \ge 0$. For $x \in \mathbb{A}^\omega$ we let $\mathrm{Fact}(x) = \{x_ix_{i+1} \cdots x_{i+j} \mid i, j \ge 0\}$ denote the set of (non-empty) factors of x. A word $x \in \mathbb{A}^\omega$ is said to be (purely) *periodic* if $x = u^\omega$, $u \in \mathbb{A}^+$, and *ultimately periodic* if some suffix of x is periodic. A word x is called *aperiodic* if it is not ultimately periodic.

Let $\varphi : \mathbb{A}^+ \to C$ be any mapping of $\mathbb{A}^+$ into a finite non-empty set C. We call the elements of C *colors* and φ a *finite coloring* of $\mathbb{A}^+$. We consider three general notions of monochromatic factorization of x relative to the coloring φ.

Definition 1. Let $\varphi : \mathbb{A}^+ \to C$ be a finite coloring of $\mathbb{A}^+$ and $x \in \mathbb{A}^\omega$. A factorization $x = V_0V_1V_2 \cdots$ with each $V_i \in \mathbb{A}^+$ is called:

- *φ-monochromatic* if $\exists c \in C$ such that $\varphi(V_i) = c$ for all $i \ge 0$.
- *φ-sequentially monochromatic* if $\exists c \in C$ such that $\varphi(V_iV_{i+1} \cdots V_{i+j}) = c$ for all $i, j \ge 0$.
- *φ-ultra monochromatic* if $\exists c \in C$ such that for all $k \ge 1$, and all $0 \le n_1 < n_2 < \cdots < n_k$, and all permutations σ of $\{1, 2, \ldots, k\}$ we have $\varphi(V_{n_{\sigma(1)}}V_{n_{\sigma(2)}} \cdots V_{n_{\sigma(k)}}) = c$.

Clearly any φ-ultra monochromatic factorization is φ-sequentially monochromatic and any φ-sequentially monochromatic factorization is φ-monochromatic. We begin with some examples.

Let $\varphi : \mathbb{A}^+ \to C$ be any finite coloring, and let $x = u^\omega$, $u \in \mathbb{A}^+$, be a periodic infinite word. Then the factorization $x = u \cdot u \cdot u \cdots$ is φ-monochromatic. In general this factorization need not be φ-sequentially monochromatic.

Let $\mathbb{T} = t_0t_1t_2 \cdots \in \{0, 1\}^\omega$ denote the *Thue–Morse infinite word*, where t_n is defined as the sum modulo 2 of the digits in the binary expansion of n.

$$\mathbb{T} = 0110100110010110100 10 \cdots.$$

The origins of $\mathbb{T}$ go back to the beginning of the last century with the works of A. Thue [17,18] in which he proves amongst other things that $\mathbb{T}$ is *overlap-free* i.e., contains no word of the form uuu' where u' is a non-empty prefix of u.

Consider $\varphi : \{0, 1\}^+ \to \{0, 1\}$ defined by $\varphi(u) = 0$ if u is a prefix of $\mathbb{T}$ and $\varphi(u) = 1$ otherwise. It is easy to see that $\mathbb{T}$ may be factored uniquely as $\mathbb{T} = V_0V_1V_2 \cdots$ where each $V_i \in \{0, 01, 011\}$. Since each V_i is a prefix of $\mathbb{T}$, it follows that this factorization is φ-monochromatic. Since $V_1V_2 = 010$ is not a prefix of $\mathbb{T}$, this factorization of $\mathbb{T}$ is not φ-sequentially monochromatic. Next consider the coloring $\varphi' : \{0, 1\}^+ \to \{0, 1, 2\}$ defined