Journal of Combinatorial Theory, Series A 137 (2016) 207–225

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Chains in weak order posets associated to involutions

Mahir Bilen Can [a], Michael Joyce [a], Benjamin Wyser [b]

[a] *Department of Mathematics, Tulane University, New Orleans, LA 70118, USA*
[b] *Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*

A R T I C L E   I N F O

Article history:
Received 16 September 2014
Available online 14 September 2015

Keywords:
Involutions
Weak order
Borel orbits
Symmetric varieties

A B S T R A C T

The $\mathcal{W}$-set of an element of a weak order poset is useful in the cohomological study of the closures of spherical subgroups in generalized flag varieties. We explicitly describe in a purely combinatorial manner the $\mathcal{W}$-sets of three different weak order posets: the set of all involutions in the symmetric group, the set of fixed point free involutions of the symmetric group, and the set of charged involutions. These distinguished sets of involutions parametrize Borel orbits in the classical symmetric spaces associated to the general linear group. In particular, we characterize the maximal chains of an arbitrary lower order ideal in any of these three posets.

1. Introduction

Given a reductive group G, an algebraic variety X equipped with a G-action is said to be spherical if a Borel subgroup B of G has a dense orbit in X. (All varieties in this paper are defined over an algebraically closed field of characteristic $\neq 2$.) A subgroup

E-mail addresses: mcan@tulane.edu (M.B. Can), mjoyce3@tulane.edu (M. Joyce), bwyser@illinois.edu (B. Wyser).

http://dx.doi.org/10.1016/j.jcta.2015.09.001
0097-3165/

$H \subseteq G$ is called spherical if the homogeneous space G/H is spherical. The geometry of spherical varieties provides a rich source of combinatorial structures.

A spherical G-variety X always has finitely many B-orbits [1,7,12]. We denote the set of B-orbit closures in X by $\mathcal{B}(X)$. The set $\mathcal{B}(X)$ possesses two geometrically natural partial orders: the Bruhat order (given by inclusion), and the weak order. The *weak order* is the transitive closure of the covers given by $Y_1 \lessdot Y_2$ if and only if $Y_2 = PY_1 \neq Y_1$ for some minimal parabolic subgroup P containing B. Not much is known about these poset structures for arbitrary spherical varieties – a parametrization of the B-orbits is unknown in general. Yet, in the special case of symmetric homogeneous spaces, there is a natural order-preserving map from the weak order poset $\mathcal{B}(X)$ to a weak order poset of twisted involutions of the Weyl group W of G [10]. Parameterizations of $\mathcal{B}(X)$ for all classical type symmetric homogeneous spaces are given in [6]. See [11] and [5] for more on the combinatorics of Borel orbits of symmetric varieties.

In this paper, we study maximal chains in the weak order on three sets of involutions:

(1) The poset of all involutions in the symmetric group S_n.
(2) The poset of all fixed-point free involutions in S_n.
(3) The poset of all “charged involutions” of S_n. These are defined in Section 2.5.

These posets are the opposite of weak order posets $\mathcal{B}(X)$ for three classical symmetric homogeneous spaces G/H of type A, where $G = \mathrm{GL}_n$, and H is, respectively, the central extension of the orthogonal group O_n, the central extension of the symplectic group Sp_n, and the product subgroup $\mathrm{GL}_p \times \mathrm{GL}_q$. We combinatorially characterize chains in the three weak order posets of involutions listed above. Such descriptions are useful for understanding the stratification of H-orbits in G/B, or equivalently of B-orbits in G/H and constitute one of the fundamental problems in the study of spherical varieties. While our results have geometric significance, we emphasize that the results and methods in this paper are purely combinatorial in nature and do not rely on any geometric considerations. The results obtained here will be combined with geometric arguments to obtain new Schubert polynomial identities in a future work [4].

Our primary combinatorial object of study is the $\mathcal{W}$-set of an element of one of our three posets. The notion of a $\mathcal{W}$-set is introduced by Brion in [2] in a geometric context. Here, we give a purely combinatorial definition. Let P be a poset with each cover $\lessdot$ assigned a set of labels from the set $\{1, 2, \dots, n-1\}$ for some integer n. If $p \lessdot p'$ and j belongs to the set of labels for $\lessdot$, write $p \lessdot_j p'$. Assuming P to have a unique minimal element $\hat{0}$, the *$\mathcal{W}$-set* of an element $p \in P$, denoted $\mathcal{W}(p)$, consists of all $w = s_{j_\ell} \cdots s_{j_2} s_{j_1} \in S_n$ of length ℓ such that

$$\hat{0} = p_0 \lessdot_{j_1} p_1 \lessdot_{j_2} \cdots \lessdot_{j_\ell} p_\ell = p,$$

for some $p_1, p_2, \dots, p_{\ell-1} \in P$. (As usual, s_j denotes the simple transposition of S_n which interchanges j and $j+1$.)