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Simple recurrence formulas to count maps on orientable surfaces

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A B S T R A C T

We establish a simple recurrence formula for the number Q_g^n of rooted orientable maps counted by edges and genus. We also give a weighted variant for the generating polynomial $Q_g^n(x)$ where x is a parameter taking the number of faces of the map into account, or equivalently a simple recurrence formula for the refined numbers $M_g^{i,j}$ that count maps by genus, vertices, and faces. These formulas give by far the fastest known way of computing these numbers, or the fixed-genus generating functions, especially for large g. In the very particular case of one-face maps, we recover the Harer–Zagier recurrence formula.

Our main formula is a consequence of the KP equation for the generating function of bipartite maps, coupled with a Tutte equation, and it was apparently unnoticed before. It is similar in appearance to the one discovered by Goulden and Jackson for triangulations, and indeed our method to go from the KP equation to the recurrence formula can be seen as a combinatorial simplification of Goulden and Jackson's approach (together with one additional combinatorial trick).

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All these formulas have a very combinatorial flavour, but finding a bijective interpretation is currently unsolved.

1. Introduction and main results

A *map* is a connected graph embedded in a compact connected orientable surface in such a way that the regions delimited by the graph, called *faces*, are homeomorphic to open discs. Loops and multiple edges are allowed. A *rooted* map is a map in which an angular sector incident to a vertex is distinguished, and the latter is called the *root vertex*. The *root edge* is the edge encountered when traversing the distinguished angular sector clockwise around the root vertex. Rooted maps are considered up to oriented homeomorphisms preserving the root sector.

A map is *bipartite* if its vertices can be coloured with two colours, say black and white, in such a way that each edge links a white and a black vertex. Unless otherwise mentioned, bipartite maps will be endowed with their *canonical bicolouration* in which the root vertex is coloured white. The *degree* of a face in a map is equal to the number of edge sides along its boundary, counted with multiplicity. Note that in a bipartite map every face has even degree, since colours alternate along its boundary.

A *quadrangulation* is a map in which every face has degree 4. There is a classical bijection, that goes back to Tutte [26], between bipartite quadrangulations with n faces and genus g, and rooted maps with n edges and genus g. It is illustrated on Fig. 1. This bijection transports the number of faces of the map to the number of white vertices of the quadrangulation (in the canonical bicolouration).

For $g, n \geq 0$, we let Q_g^n be the number of rooted bipartite quadrangulations of genus g with n faces. Equivalently, by Tutte's construction, Q_g^n is the number of rooted maps of genus g with n edges. By convention we admit a single map with no edges and which has genus zero, one face, and one vertex. Our first result is the following recurrence formula:

Theorem 1. *The number Q_g^n of rooted maps of genus g with n edges (which is also the number of rooted bipartite quadrangulations of genus g with n faces) satisfies the following recurrence relation:*

$$\frac{n+1}{6}Q_g^n = \frac{4n-2}{3}Q_g^{n-1} + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2} + \frac{1}{2}\sum_{\substack{k+\ell=n \\ k,\ell\geq 1}}\sum_{\substack{i+j=g \\ i,j\geq 0}}(2k-1)(2\ell-1)Q_i^{k-1}Q_j^{\ell-1},$$

for $n \geq 1$, with the initial conditions $Q_g^0 = \mathbf{1}_{\{g=0\}}$, and $Q_g^n = 0$ if $g < 0$ or $n < 0$.