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Wreath determinants for group–subgroup pairs

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ABSTRACT

The aim of the present paper is to generalize the notion of the group determinants for finite groups. For a finite group G and its subgroup H, one may define a rectangular matrix of size $\#H \times \#G$ by $X = \big(x_{hg^{-1}}\big)_{h\in H, g\in G}$, where $\{x_g \mid g \in G\}$ are indeterminates indexed by the elements in G. Then, we define an invariant $\Theta(G,H)$ for a given pair (G,H) by the k-wreath determinant of the matrix X, where k is the index of H in G. The k-wreath determinant of an n by kn matrix is a relative invariant of the left action by the general linear group of order n and of the right action by the wreath product of two symmetric groups of order k and n. Since the definition of $\Theta(G,H)$ is *ordering-sensitive*, the representation theory of symmetric groups is naturally involved. When G is abelian, if we specialize the indeterminates to powers of another variable q suitably, then $\Theta(G,H)$ factors into the product of a power of q and polynomials of the form $1 - q^r$

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for various positive integers r. We also give examples for non-abelian group–subgroup pairs.

1. Introduction

It is Frobenius who initiated the character theory of finite groups [2]. At the very first stage of his study, the *group determinant* $\Theta(G)$ of a given finite group G, which is defined as the determinant

$$\Theta(G) := \det\bigl(x_{uv^{-1}}\bigr)_{u,v\in G} \tag{1}$$

of the group matrix $\bigl(x_{uv^{-1}}\bigr)_{u,v\in G}$, played an important role. Here $\{x_g \mid g \in G\}$ are indeterminates indexed by the elements in G. (One should note that the definition of $\Theta(G)$ is independent of the choice of the ordering of elements in G.) Indeed, the group determinant $\Theta(G)$ reflects the structure of the regular representation of G, which contains all the equivalence classes of the irreducible representations of G. The factorization of $\Theta(G)$ corresponds to the irreducible decomposition of the regular representation, and the irreducible character values appear as coefficients in the factors. In 1991, Formanek and Sibley [3] showed that two groups are isomorphic if and only if their group determinants coincide under a suitable correspondence between the sets of indeterminates for these groups:

$$\Theta(G) = \Theta(G') \iff G \cong G'. \tag{2}$$

Namely, the group determinant is a perfect invariant for finite groups.

Let H be a subgroup of a finite group G, set $n := \#H$, and $k := \#G/H$ denotes the index of H in G. In this paper, we extend the notion of group determinants. Actually, we define an invariant $\Theta(G, H)$ for the pair (G, H), G being a finite group and H its subgroup, by employing the *wreath determinant* [5]. For a positive integer k, the *k-wreath determinant* wrdet_k is a polynomial function on the set of n by kn matrices for each positive integer n characterized by (i) multilinearity in column vectors, (ii) relative GL_n-invariance from the left, and (iii) $\mathfrak{S}_k^n$-invariance with respect to permutations in columns, $\mathfrak{S}_k$ being the symmetric group of order k (see Section 2.1 for the precise definition). Roughly, $\Theta(G, H)$ is defined to be

$$\Theta(G, H) := \mathrm{wrdet}_k\bigl(x_{hg^{-1}}\bigr)_{\substack{h\in H\\ g\in G}}.$$

In fact, since wrdet_k is *not* a relative invariant under general permutations in columns (i.e. the action of $\mathfrak{S}_{kn}$ from the right), we should take account of the *ordering* of G to define $\Theta(G, H)$. This is a crucial difference from $\Theta(G)$. We note that $\Theta(G, G)$ is nothing but the original group determinant $\Theta(G)$ since the 1-wreath determinant is the ordinary determinant.