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Hopf algebra structure on packed square matrices

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A B S T R A C T

We construct a new bigraded Hopf algebra whose bases are indexed by square matrices with entries in the alphabet $\{0, 1, \ldots, k\}$, $k \geqslant 1$, without null rows or columns. This Hopf algebra generalizes the one of permutations of Malvenuto and Reutenauer, the one of k-colored permutations of Novelli and Thibon, and the one of uniform block permutations of Aguiar and Orellana. We study the algebraic structure of our Hopf algebra and show, by exhibiting multiplicative bases, that it is free. We moreover show that it is self-dual and admits a bidendriform bialgebra structure. Besides, as a Hopf subalgebra, we obtain a new one indexed by alternating sign matrices. We study some of its properties and algebraic quotients defined through alternating sign matrices statistics.

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0. Introduction

The combinatorial class of permutations is naturally endowed with two operations. One of them, called *shifted shuffle product*, takes two permutations as input and puts these together by blending their letters. The other one, called *deconcatenation coproduct*, takes one permutation as input and takes it apart by cutting it into prefixes and suffixes. These two operations satisfy certain compatibility relations, resulting in that the vector space spanned by the set of permutations forms a Hopf algebra [26], namely the *Malvenuto–Reutenauer Hopf algebra*, also known as **FQSym** [6].