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A combinatorial formula for principal minors of a matrix with tree-metric exponents and its applications

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ABSTRACT

Let T be a tree with a vertex set $\{1, 2, \dots, N\}$. Denote by d_{ij} the distance between vertices i and j. In this paper, we present an explicit combinatorial formula of principal minors of the matrix $(t^{d_{ij}})$, and its applications to tropical geometry, study of multivariate stable polynomials, and representation of valuated matroids. We also give an analogous formula for a skew-symmetric matrix associated with T.

1. Introduction

Let $T = (V, E)$ be a tree, where $V = \{1, 2, \dots, N\}$. For $i, j \in V$, denote by d_{ij} the number of edges of the unique path between i and j in T. With an indeterminate t, define the matrix $A = (a_{ij})$ by

$$a_{ij} := t^{d_{ij}} \quad (i, j \in V).$$

This matrix appeared in the study of the q-distance matrix of a tree [2]. Yan and Yeh [27] showed that $\det A$ is given by the following simple formula:

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Theorem 1.1. *(See Yan and Yeh [27].)* $\det A = (1 - t^2)^{N-1}$.

Our main result can be understood as an extension of Yan–Yeh's formula to principal minors of A. The motivation of our investigation, however, comes from study of multivariate stable polynomials [6,7,9], tropical geometry [11,25], and representation of valuated matroids [13,15]. To state our result, let us introduce some notions. For $X \subseteq V$, denote by $A[X]$ the principal submatrix of A consisting of a_{ij} for $i, j \in X$. We say that a forest $F = (V_F, E_F)$ *is spanned by* X if $X \subseteq V_F$ and all leaves of F are contained in X. Note that the subtree of T spanned by X is the unique minimal subtree including X, which is denoted by $T_X = (V_X, E_X)$. Define $c(F)$ as the number of connected components of F. Denote by $\deg_F(v)$ the degree of a vertex v in F. Then our main result is the following:

Theorem 1.2.

$$\det A[X] = \sum_F (-1)^{|X|+c(F)} t^{2|E_F|} \prod_{v \in V_F \setminus X} (\deg_F(v) - 1), \tag{1.1}$$

where the sum is taken over all subgraphs F of T spanned by X. In particular, the leading term is given by

$$(-1)^{|X|+1} t^{2|E_X|} \prod_{v \in V_X \setminus X} (\deg_{T_X}(v) - 1). \tag{1.2}$$

In the case $X = V$, the formula (1.1) coincides with the binomial expansion of Yan–Yah's formula.

Our formula brings a strong consequence on the signature of $A[X]$. Recall that the signature of a symmetric matrix is a pair (p, q) of the number p of positive eigenvalues and the number q of negative eigenvalues. When we substitute a sufficiently large value for t, the sign of $\det A[X]$ is determined by the leading term. By (1.2), $\det A[X] > 0$ if $|X|$ is odd, and $\det A[X] < 0$ if $|X|$ is even. From Sylvester's law of inertia, the number of sign changes of leading principal minors is equal to the number of negative eigenvalues (see [17, Theorem 2 in Chapter X]). Therefore the signature of $A[X]$ is $(1, |X| - 1)$. This argument works on the field $\mathbf{R}\{t\}$ of Puiseux series (defined in Section 2). Thus we have the following.

Corollary 1.3. *The signature of $A[X]$ is $(1, |X| - 1)$.*

In particular, $A[X]$ is nonsingular and defines the Minkowski inner product, i.e., a nondegenerate bilinear form with exactly one positive eigenvalue.

We also consider a skew-symmetric matrix $B = (b_{ij})$ defined by

$$b_{ij} = -b_{ji} = t^{d_{ij}} \quad (i < j).$$