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Interlacing networks: Birational RSK, the octahedron recurrence, and Schur function identities

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ABSTRACT

Motivated by the problem of giving a bijective proof of the fact that the birational RSK correspondence satisfies the octahedron recurrence, we define interlacing networks, which are certain planar directed networks with a rigid structure of sources and sinks. We describe an involution that swaps paths in these networks and leads to Plücker-like three-term relations among path weights. We show that indeed these relations follow from the Plücker relations in the Grassmannian together with some simple rank properties of the matrices corresponding to our interlacing networks. The space of matrices obeying these rank properties forms the closure of a cell in the matroid stratification of the totally nonnegative Grassmannian. Not only does the octahedron recurrence for RSK follow immediately from the three-term relations for interlacing networks, but also these relations imply some interesting identities of Schur functions reminiscent of those obtained by Fulmek and Kleber. These Schur function identities lead to some results on Schur positivity for expressions of the form $s_\nu s_\rho - s_\lambda s_\mu$.

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1. Introduction

The Robinson–Schensted (RS) correspondence is a bijection between permutations σ in the symmetric group S_n and pairs (P, Q) of standard Young tableaux of the same shape $\lambda \vdash n$. Under this correspondence, the first part λ_1 of λ has a simple interpretation as the size of the longest increasing subsequence of σ; more generally, the partial sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ is equal to the maximum size of a union of k disjoint increasing subsequences in σ. This description of the shape under RS of a permutation in terms of longest increasing subsequences is known as Greene's theorem [12]. The Robinson–Schensted–Knuth (RSK) correspondence is a generalization of the RS correspondence which takes arbitrary $n \times n$ $\mathbb{N}$-matrices A to pairs (P, Q) of semistandard Young tableaux of the same shape λ. In the RSK correspondence, the first part λ_1 has an analogous interpretation as the maximum weight of a path in A from $(1, 1)$ to (n, n). Here the weight of a path is just the sum of the entries of the boxes it visits. Similarly, the partial sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ is equal to the maximum weight over k-tuples of noncrossing paths in A connecting $(1, 1), (1, 2), \ldots, (1, k)$ to $(n, n - k + 1), (n, n - k + 2), \ldots, (n, n)$. This extension of Greene's theorem to RSK is apparently folklore; the best reference we have for it is [18, Theorem 12]. For general background on RSK and its importance in the theory of symmetric functions, see [32, Chapter 7].

Recently there has been significant interest in a birational lifting of RSK which takes matrices with entries in $\mathbb{R}_{>0}$ to certain three-dimensional arrays with entries in $\mathbb{R}_{>0}$; see, e.g., [17,23,5,7,3,25,24]. The noncrossing paths interpretation of RSK has a direct analogue in the birational setting (where in this process of detropicalization, maximums become sums and sums become products). Also it turns out that, subject to the proper renormalization, the output array of this birational map obeys the octahedron recurrence [6,7]. For an excellent introduction to the octahedron recurrence and its appearance in various combinatorial problems, see [31]. This paper's main motivation is to provide a combinatorial proof of the fact that the sums over weighted tuples of noncrossing paths of the form encountered in the birational RSK correspondence obey the octahedron recurrence. Although this has been established already in [7] by algebraic means, we present a direct, bijective proof, akin to the standard proof of the famous Lindström–Gessel–Viennot lemma (see [33, Theorem 2.7.1]).

To this end, in Section 2 we define "interlacing networks", which are certain planar directed networks with a rigid structure of sources and sinks. In Section 3 we state and prove our main result concerning these networks: the existence of an involution that swaps pairs of tuples of noncrossing paths connecting sinks and sources. This involution leads to several three-term relations between the weights of these pairs of tuples of noncrossing paths that are akin to the three-term Plücker relations. Because of the well-known correspondence between totally-positive matrices and planar directed networks, these three-term relations are equivalent to determinantal identities for matrices of a particular form. We study these matrices in Section 4, where we give an alternative algebraic proof of these determinantal identities via the Plücker relations. Indeed, the