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A truncated Jacobi triple product theorem

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A R T I C L E I N F O

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A B S T R A C T

Recently, G.E. Andrews and M. Merca considered a truncated version of Euler's pentagonal number theorem and obtained a nonnegativity result. They asked the same question on a truncated Jacobi triple product identity, which can be found as a conjecture in a paper of V.J.W. Guo and J. Zeng. In this paper, we provide an answer to the question, which is purely combinatorial. We also provide a combinatorial proof of the main theorem in the paper of Andrews and Merca.

1. Introduction

One of the well-known theorems in the partition theory is the following pentagonal number theorem.

Theorem 1.1 *(Euler's pentagonal number theorem). We have*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty .$$

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This identity leads to

$$\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = 1. \tag{1.1}$$

Here and throughout this paper, we use the following customary q-series notation:

$$\begin{aligned}
(a;q)_0 &:= 1,\\
(a;q)_\infty &:= \prod_{k=0}^{\infty}(1-aq^k),\\
(a;q)_n &:= \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad \text{for any } n,\\
\begin{bmatrix} n\\ k\end{bmatrix} &:= \begin{bmatrix} n\\ k\end{bmatrix}_q := \begin{cases} 0, & \text{if } k<0 \text{ or } k>n,\\ \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{otherwise.}\end{cases}
\end{aligned}$$

Recently, G.E. Andrews and M. Merca considered a truncated version of (1.1) and obtained the following result:

$$\begin{aligned}
&\frac{1}{(q;q)_\infty}\sum_{n=0}^{k-1}(-1)^n q^{n(3n+1)/2}(1-q^{2n+1})\\
&\quad = 1+(-1)^{k-1}\sum_{n=1}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix} n-1\\ k-1\end{bmatrix},
\end{aligned} \tag{1.2}$$

from which they deduced the following partition theorem.

Theorem 1.2. *(See [4, Theorem 1.1].) For $n>0$, $k\ge 1$*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\big(p\big(n-j(3j+1)/2\big)-p\big(n-j(3j+5)/2-1\big)\big)=M_k(n)$$

where $M_k(n)$ is the number of partitions of n in which k is the least integer that is not a part and there are more parts $>k$ than there are $<k$.

Theorem 1.1 has the following generalization.

Theorem 1.3 *(The Jacobi triple product identity). For $z\neq 0$,*

$$\sum_{n=-\infty}^{\infty}(-z)^n q^{\binom{n}{2}} = (z;q)_\infty(q/z;q)_\infty(q;q)_\infty.$$