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Proofs of two conjectures on truncated series

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ABSTRACT

In this paper, we prove two conjectures on truncated series. The first conjecture made by G.E. Andrews and M. Merca is related to Jacobi's triple product identity, while the second conjecture by V.J.W. Guo and J. Zeng is related to Jacobi's identity.

1. Introduction

Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{j=0}^{\infty} (-1)^j q^{j(3j-1)/2} \left(1 - q^{2j+1}\right) \tag{1.1}$$

plays an important role in the theory of partitions. It is closely related to the partition function $p(n)$ [1]. In Eq. (1.1) and for the rest of this article, we use the notations

$$(x_1, x_2, \dots, x_k; q)_m := \prod_{n=0}^{m-1} \left(1 - x_1 q^n\right)\left(1 - x_2 q^n\right) \cdots \left(1 - x_k q^n\right),$$

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$$(x_1, x_2, \dots, x_k; q)_\infty := \prod_{n=0}^{\infty} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$

and we require $|q| < 1$ for absolute convergence.

Recently Andrews and Merca proved a new expansion for partial sums of Euler's pentagonal number series (see [3, Lemma 1.2]). This gives the inequality, for $k \geq 1$,

$$(-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \big(p\big(n - j(3j+1)/2\big) - p\big(n - j(3j+5)/2 - 1\big)\big) \geq 0 \tag{1.2}$$

with strict inequality if $n \geq k(3k+1)/2$. Using certain partition pairs, Louis W. Kolitsch and M. Burnette interpreted the partial sum on the left side (see [8, Corollary 2]). The special case of (1.2) with $k = 2$ was first proved by Merca [11]. It is easy to verify that (1.2) is equivalent to the result that, for $k \geq 1$, the truncated pentagonal number series

$$(-1)^{k-1} \frac{\sum_{j=0}^{k-1} (-1)^j q^{j(3j-1)/2} (1 - q^{2j+1})}{(q; q)_\infty} \tag{1.3}$$

has nonnegative coefficients. At the end of their paper (see Problem **2**), Andrews and Merca [3] considered a generalization of (1.3) and conjectured the following.

Conjecture 1.1 *(Andrews and Merca). For positive integers k, R, S with $k \geq 1$ and $1 \leq S < R/2$, the coefficient of q^n with $n \geq 1$ in*

$$(-1)^{k-1} \frac{\sum_{j=0}^{k-1} (-1)^j q^{Rj(j+1)/2 - Sj} (1 - q^{(2j+1)S})}{(q^R, q^{R-S}, q^R; q^R)_\infty}$$

is nonnegative.

Three remarks.

(1) Setting $R = 3$ and $S = 1$ in the series in Conjecture 1.1 we recover (1.3).

(2) We note that the series in Conjecture 1.1 comes from the following special case of Jacobi's triple product identity

$$\left(q^R, q^{R-S}, q^R; q^R\right)_\infty = \sum_{j=0}^{\infty} (-1)^j q^{Rj(j+1)/2 - Sj} \left(1 - q^{(2j+1)S}\right). \tag{1.4}$$

Setting $R = 3$ and $S = 1$ in (1.4), we recover (1.1).

(3) An equivalent version of Conjecture 1.1 was also given by Guo and Zeng (see [7, Conjecture 6.1]).

We state the first result of this paper in the following theorem.