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Signed excedance enumeration in classical and affine Weyl groups

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ABSTRACT

Based on the notions of colored and absolute excedances introduced by Bagno and Garber and their affine versions introduced by Mongelli, we compute the signed generating function of such statistics. Moreover, whenever possible, we derive a combinatorial interpretation of the coefficients of such generating functions. This paper is inspired by a paper of S. Sivasubramanian in which the author enumerates signed statistics on the group of classical permutations.

0. Introduction

The symmetric group S_n has many interesting permutation statistics. The most important ones are probably inversions, descents, excedances and the major index. It is well-known that the descent statistic and the excedance statistic are equidistributed, and the same is true for the inversion statistic and the major index. Some of these statistics, namely inversions and descents, have natural analogues in all Coxeter groups. Generalizing the other two and the corresponding results has been much more difficult and, already for the group of signed permutations S_n^B (also known as the hyperoctahedral group), has occupied a number of mathematicians for a number of years

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(see e.g. [3,6,8,7,13,14]). Recently, the excedance statistic was generalized also to the affine Weyl groups $\widetilde{A}_n$, $\widetilde{C}_n$, $\widetilde{B}_n$ and $\widetilde{D}_n$ (see [9,17]).

Several signed enumeration results over permutations are known (see e.g. [15,11,23, 19,20,1,14]). Recently, Sivasubramanian [21] proved results on signed-excedance enumeration in the set of classical permutations and its subset of all derangements. The main idea used by the author is to compute the signed generating functions of the statistics via determinants.

The purpose of this paper is to extend the study of signed-excedance statistic done by Sivasubramanian to all finite and affine Weyl groups and to all derangements of such groups.

The organization of this paper is as follows. In the next section we collect several definitions, notation and results that will be used in the sequel. In Section 2 we study properties of the signed-excedance statistics in the finite Weyl groups B_n and D_n. In Section 3 we compute the signed generating function for the affine groups $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$. In some cases we give combinatorial interpretations to the coefficients of such generating functions.

1. Notation, definitions and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbb{P} := \{1, 2, 3, \ldots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$, $\mathbb{Z}$ be the ring of integers, $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ and $\mathbb{R}$ be the field of real numbers. Given $n, m \in \mathbb{Z}$, $n \leq m$, we let $[n, m] := \{n, n+1, \ldots, m\}$. The cardinality of a set A will be denoted by $|A|$. We denote by χ the indicator function.

Given a non-negative integer i, its q-analogue is defined as $[i]_q = 1+q+q^2+\cdots+q^{i-1}$, where q is an indeterminate and $[0]_q = 0$.

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we say that an index $i \in [1, n]$ is an *excedance* of σ if $a_i > i$. We denote by $\mathrm{exc}(\sigma)$ the number of excedances of σ. We say that the pair (i, j) is an *inversion* of σ if $i < j$ and $a_i > a_j$. We denote by $\mathrm{inv}(\sigma)$ the number of inversions of σ. Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n := S([1, n])$. If $\sigma \in S_n$ then we write $\sigma = [\sigma_1, \ldots, \sigma_n]$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. We call such sequence the window notation of σ to distinguish it from the disjoint cycle notation (see e.g. [22, p. 17]). It is well-known that S_n is generated by $s_1, \ldots, s_{n-1}$ where $s_i = [1, \ldots, i-1, i+1, i, \ldots, n]$ for all $i \in [1, n-1]$; moreover, if we write a permutation σ as a product of generators, then the minimal number of factors used in any such product is equal to $\mathrm{inv}(\sigma)$ (in Coxeter theoretic language it is known as the *length* of σ).

In [15] and [21] the following result is shown.

Theorem 1.1. *For $n \geq 1$, the signed generating function of the excedance statistic on S_n is given by $\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}(\sigma)} = (1-q)^{n-1}$.*