J. Combin. Theory Ser. A 131 (2015) 153–186

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Results and conjectures on the number of standard strong marked tableaux

Susanna Fishel [a,1], Matjaž Konvalinka [b,2]

[a] School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ, USA
[b] Department of Mathematics, University of Ljubljana, Slovenia

ARTICLE INFO

Article history:
Received 31 August 2013
Available online 9 December 2014

Keywords:
k-Schur functions
Strong marked tableaux
Enumeration

ABSTRACT

Many results involving Schur functions have analogues involving k-Schur functions. Standard strong marked tableaux play a role for k-Schur functions similar to the role standard Young tableaux play for Schur functions. We discuss results and conjectures toward an analogue of the hook-length formula.
New tools, residue and quotient tables, are presented, which allow for efficient computation of strong covers and have the potential to describe many other phenomena in k-function theory.

1. Introduction

In 1988, Macdonald [17] introduced a new class of symmetric functions and conjectured that they expand positively in terms of Schur functions. This conjecture, verified in [7], has led to an enormous amount of work, including the development of the k-Schur

E-mail addresses: sfishel1@asu.edu (S. Fishel), matjaz.konvalinka@fmf.uni-lj.si (M. Konvalinka).
URLs: http://math.la.asu.edu/~fishel/ (S. Fishel), http://www.fmf.uni-lj.si/~konvalinka/ (M. Konvalinka).

[1] Susanna Fishel was partially supported by NSF grant 1200280 and Simons Foundation grant 209806.
[2] Matjaž Konvalinka was partially supported by Research Program L1-069 of the Slovenian Research Agency.

http://dx.doi.org/10.1016/j.jcta.2014.11.011

functions. The k-Schur functions were defined in [13]. Lascoux, Lapointe, and Morse conjectured that they form a basis for a certain subspace of the space of symmetric functions and that the Macdonald polynomials indexed by partitions whose first part is not larger than k expand positively in terms of the k-Schur functions, leading to a refinement of the Macdonald conjecture. The k-Schur functions have since been found to arise in other contexts; for example, as Schubert classes in the quantum cohomology of the Grassmannian [16], and, more generally, in the cohomology of the affine Grassmannian [9].

One of the intriguing features of standard Young tableaux is the Frame–Thrall–Robinson hook-length formula, which enumerates them. It has many different proofs and many generalizations, see e.g. [20, Chapter 7], [5,2] and the references therein.

In this paper, we partially succeed in finding an analogue of the hook-length formula for standard strong marked tableaux (or starred tableaux for short), which are a natural generalization of standard Young tableaux in the context of k-Schur functions. For a fixed n, the shape of a starred tableau (see Section 2.6 for a definition) is necessarily an n-core, a partition for which all hook-lengths are different from n. In [12], a formula is given for the number of starred tableaux for $n = 3$.

Proposition 1.1. *(See [12, Proposition 9.17].) For a 3-core λ, the number of starred tableaux of shape λ equals*

$$\frac{m!}{2^{\lfloor \frac{m}{2} \rfloor}},$$

where m is the number of boxes of λ with hook-length < 3. □

The number of 2-hooks is $\lfloor \frac{m}{2} \rfloor$. Therefore we can rewrite the result as

$$\frac{m!}{\prod_{\substack{i,j\in\lambda \\ h_{ij}<3}} h_{ij}}.$$

Note that this is reminiscent of the classical hook-length formula.

The authors left the enumeration for $n > 3$ as an open problem. The main result (Theorem 3.1) of this paper implies the existence, for each n, of $(n-1)!$ rational numbers which we call *correction factors*. Once the corrections factors have been calculated by enumerating all starred tableaux for certain shapes, the number of starred tableaux of shape λ for any n-core λ can be easily computed. In fact, Theorem 3.1 is a t-analogue of the hook formula. The theorem is “incomplete” in the sense that we were not able to find explicit formulas for the (weighted) correction factors. We have, however, been able to state some of their properties (some conjecturally), the most interesting of these properties being unimodality (Conjecture 3.7).

Another result of interest is a new, alternative description of strong marked covers via simple triangular arrays of integers which we call *residue tables* and *quotient tables* (see Theorem 5.2). Many other results in the theory of k-bounded partitions and k-Schur