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An extremal problem on crossing vectors

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For positive integers w and k, two vectors A and B from $\mathbb{Z}^w$ are called *k-crossing* if there are two coordinates i and j such that $A[i] - B[i] \geqslant k$ and $B[j] - A[j] \geqslant k$. What is the maximum size of a family of pairwise 1-crossing and pairwise non-k-crossing vectors in $\mathbb{Z}^w$? We state a conjecture that the answer is k^{w-1}. We prove the conjecture for $w \leqslant 3$ and provide weaker upper bounds for $w \geqslant 4$. Also, for all k and w, we construct several quite different examples of families of desired size k^{w-1}. This research is motivated by a natural question concerning the width of the lattice of maximum antichains of a partially ordered set.

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1. Introduction

We deal with vectors in $\mathbb{Z}^w$, which we call just *vectors*. The ith coordinate of a vector $A \in \mathbb{Z}^w$ is denoted by $A[i]$, for $1 \leqslant i \leqslant w$. The product ordering on $\mathbb{Z}^w$ is defined by setting $A \leqslant B$ for $A, B \in \mathbb{Z}^w$ whenever $A[i] \leqslant B[i]$ for every coordinate i. When $k \geqslant 1$, we say that vectors A and B from $\mathbb{Z}^w$ are *k-crossing* if there are coordinates i and j for which $A[i] - B[i] \geqslant k$ and $B[j] - A[j] \geqslant k$. Thus $\mathcal{A}$ is an antichain in $\mathbb{Z}^w$ if and only if any two distinct vectors from $\mathcal{A}$ are 1-crossing. A family of vectors in $\mathbb{Z}^w$ is *k-crossing-free* if it contains no two k-crossing vectors.

For positive integers k and w, let $f(k, w)$ denote the maximum size of a subset of $\mathbb{Z}^w$ with any two vectors being 1-crossing but not k-crossing. In other words, $f(k, w)$ is the maximum size of a k-crossing-free antichain in $\mathbb{Z}^w$. Note that an antichain of vectors in $\mathbb{Z}^w$ with $w \geqslant 2$ without the restriction that no two vectors are k-crossing can have infinite size (e.g. $\{(k, -k): k \in \mathbb{Z}\}$ for $w = 2$). Similarly, there are infinite k-crossing-free families of vectors in $\mathbb{Z}^w$ which are not antichains (e.g. $\{(k, k): k \in \mathbb{Z}\}$ for $w = 2$).

Determining the value of $f(k, w)$ is the main focus of this paper. The following striking simple conjecture was formulated in 2010 and never published, so we state it here with the kind permission of its authors.

Conjecture 1 *(Felsner, Krawczyk, Micek). For all $k, w \geqslant 1$, we have*

$$f(k, w) = k^{w-1}.$$

At first, it is even not clear whether $f(k, w)$ is bounded for all k and w. We prove the conjecture for $1 \leqslant w \leqslant 3$ and provide lower (matching the conjectured value) and upper bounds on $f(k, w)$ for $w \geqslant 4$. Still, we are unable to resolve the conjecture in full generality.

Theorem 2. *For $1 \leqslant w \leqslant 3$ and $k \geqslant 1$, we have*

$$f(k, w) = k^{w-1}.$$

Theorem 3. *For $w \geqslant 4$ and $k \geqslant 1$, we have*

$$k^{w-1} \leqslant f(k, w) \leqslant \min\left\{k^w - k^2(k-1)^{w-2}, \left\lceil \tfrac{w}{3} \right\rceil k^{w-1}\right\}.$$

The remainder of this paper is organized as follows. We start, in the next section, by a brief discussion of problems in partially ordered sets that initiated this research. Section 3 is devoted to the proof of Theorem 3 and the lower bound of Theorem 2. The upper bound of Theorem 2 is proved in Section 4. In Section 5, we propose another conjecture, which is at first glance more general but in fact equivalent to Conjecture 1. Concluding in Section 6, we provide examples of families witnessing $f(k, w) \geqslant k^{w-1}$ with