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A random version of Sperner's theorem

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathcal{P}(n)$ denote the power set of $[n]$, ordered by inclusion, and let $\mathcal{P}(n,p)$ be obtained from $\mathcal{P}(n)$ by selecting elements from $\mathcal{P}(n)$ independently at random with probability p. A classical result of Sperner [12] asserts that every antichain in $\mathcal{P}(n)$ has size at most that of the middle layer, $\binom{n}{\lfloor n/2 \rfloor}$. In this note we prove an analogous result for $\mathcal{P}(n,p)$: If $pn \to \infty$ then, with high probability, the size of the largest antichain in $\mathcal{P}(n,p)$ is at most $(1+o(1))p\binom{n}{\lfloor n/2 \rfloor}$. This solves a conjecture of Osthus [9] who proved the result in the case when $pn/\log n \to \infty$. Our condition on p is best-possible. In fact, we prove a more general result giving an upper bound on the size of the largest antichain for a wider range of values of p.

We write $[n]$ for the set of natural numbers up to n, and $\mathcal{P}(n)$ for the power set of $[n]$. Also, for any $0 \le k \le n$ we write $\binom{[n]}{k}$ for the subset of $\mathcal{P}(n)$ consisting of all sets of size k. A subset $\mathcal{A} \subseteq \mathcal{P}(n)$ is an *antichain* if for any $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $A = B$. So $\binom{[n]}{k}$ is an antichain for any $0 \le k \le n$; Sperner's theorem [12] states that

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in fact no antichain in $\mathcal{P}(n)$ has size larger than $\binom{n}{\lfloor n/2 \rfloor}$. Our main theorem is a random version of Sperner's theorem. For this, let $\mathcal{P}(n,p)$ be the set obtained from $\mathcal{P}(n)$ by selecting elements randomly with probability p and independently of all other choices. Write $m := \binom{n}{\lfloor n/2 \rfloor}$. Roughly speaking, our main result asserts that if $p > C/n$ for some constant C, then with high probability, the largest antichain in $\mathcal{P}(n,p)$ is approximately the same size as the 'middle layer' in $\mathcal{P}(n,p)$.

Theorem 1. *For any $\varepsilon > 0$ there exists a constant C such that if $p > C/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pm$.*

(Here, by 'with high probability' we mean with probability tending to 1 as n tends to infinity.)

The model $\mathcal{P}(n,p)$ was first investigated by Rényi [10] who determined the probability threshold for the property that $\mathcal{P}(n,p)$ is not itself an antichain, thereby answering a question of Erdős. The size of the largest antichain in $\mathcal{P}(n,p)$ for p above this threshold was first studied by Kohayakawa and Kreuter [6]. In [6] they raised the question of which values of p does the conclusion of Theorem 1 hold. Osthus [9] proved Theorem 1 in the case when $pn/\log n \to \infty$ and conjectured that this can be replaced by $pn \to \infty$. (So Theorem 1 resolves this conjecture.) Moreover, Osthus showed that, for a fixed $c > 0$, if $p = c/n$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at least $(1+o(1))(1+e^{-c/2})p\binom{n}{\lfloor n/2 \rfloor}$. So the bound on p in Theorem 1 is best-possible up to the constant C. There have also been a number of results concerning the length of (the longest) chains in $\mathcal{P}(n,p)$ and related models of random posets (see, for example, [2,7,8]).

Instead of proving Theorem 1 directly we prove the following more general result.

Theorem 2. *Let $n \in \mathbb{N}$ and $m := \binom{n}{\lfloor n/2 \rfloor}$. For any $\varepsilon > 0$ and $t \in \mathbb{N}$, there exists a constant C such that if $p > C/n^t$ then with high probability the largest antichain in $\mathcal{P}(n,p)$ has size at most $(1+\varepsilon)pmt$.*

Osthus [9] proved this result in the case when $p(n/t)^t/\log n \to \infty$. (In fact, Osthus's result allows for t to be an integer function, see [9] for the precise statement.) Moreover, Osthus showed that, for $1/n^t \ll p \ll 1/n^{t-1}$, with high probability, $\mathcal{P}(n,p)$ has an antichain of size at least $(1+o(1))pmt$ (so Theorem 2 is 'tight' in this window of p).

The method of proof of Theorem 2 also allows us to estimate the number of antichains in $\mathcal{P}(n)$ of certain fixed sizes.

Proposition 3. *Fix any $t \in \mathbb{N}$, and suppose that $m/n^t \ll s \ll m/n^{t-1}$. Then the number of antichains of size s in $\mathcal{P}(n)$ is $\binom{(t+o(1))m}{s}$.*

To prove Theorem 2, let G be the graph with vertex set $\mathcal{P}(n)$ in which distinct sets A and B are adjacent if $A \subseteq B$ or $B \subseteq A$. Then an antichain in $\mathcal{P}(n)$ is precisely an independent set in G. We follow the 'hypergraph container' approach (see, for example, [1,11]): indeed, we show that all independent sets in G are contained within a fairly small