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Degrees in oriented hypergraphs and sparse Ramsey theory

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A B S T R A C T

Let G be an r-uniform hypergraph. When is it possible to orient the edges of G in such a way that every p-set of vertices has some p-degree equal to 0? (The p-degrees generalise for sets of vertices what in-degree and out-degree are for single vertices in directed graphs.) Caro and Hansberg asked if the obvious Hall-type necessary condition is also sufficient.

Our main aim is to show that this is true for r large (for given p), but false in general. Our counterexample is based on a new technique in sparse Ramsey theory that may be of independent interest.

1. Introduction

When does a graph G have an orientation such that every out-degree is at most k? An obvious necessary condition is that $|E(H)| \leq k|V(H)|$ for every subgraph $H \subset G$. Indeed, suppose G has such an orientation and $H \subset G$. Inside H the sum of out-degrees equals the number of edges. Moreover, each vertex contributes at most k to this sum, hence the condition. Hakimi proved that this condition is in fact sufficient.

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Theorem A. *(See Hakimi [3].) Let G be a graph and $k \geq 0$ an integer. Then G has an orientation such that every vertex has out-degree at most k if and only if all subgraphs $H \subset G$ satisfy $|E(H)| \leq k|V(H)|$.*

In fact, Hakimi proved a slightly stronger result that determines all possible out-degree sequences produced by orientations of a given graph G. His proof uses induction on the number of edges, but the weaker statement given above is a straightforward consequence of Hall's marriage theorem.

What about hypergraphs? Suppose an r-uniform hypergraph G (i.e. a family of r-sets) is given an *orientation*, by which we mean that for each edge e one ordering of the vertices of e is chosen. This ordering is called the *orientation of* e. Note that each edge has exactly $r!$ possible orientations. If $r = 2$ then this coincides with the usual definition of graph orientation. We will often denote an orientation of G by $D(G)$ and the corresponding orientation of an edge e by $D(e)$.

Given an orientation $D(G)$, a vertex v and $i \in [r] = \{1, 2, \ldots, r\}$, the *$i$-degree* of v, written $d_i(v)$, is the number of edges e such that v is in the i-th position of $D(e)$. Note that if $r = 2$ then $d_1(v)$ is the out-degree and $d_2(v)$ is the in-degree of v.

When does an r-uniform hypergraph G have an orientation such that $d_1(v) \leq k$ for every vertex v? Again, an obvious necessary condition is that $|E(H)| \leq k|V(H)|$ for every subgraph $H \subset G$ (where as usual H is a subgraph of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$). Indeed, inside H the sum of $d_1(v)$ over all vertices v of H is equal to the number of edges of H, and is at most k times the number of vertices.

Caro and Hansberg showed that this condition is sufficient.

Theorem B. *(See Caro and Hansberg [1].) Let G be an r-uniform hypergraph and $k \geq 0$ an integer. Then G has an orientation such that $d_1(v) \leq k$ for all vertices v if and only if all subgraphs $H \subset G$ satisfy $|E(H)| \leq k|V(H)|$.*

They proved it by constructing a suitable maximal flow on H, and a simple proof via Hall's marriage theorem is also possible.

Now, in contrast to the situation for graphs, for oriented hypergraphs there is a sensible notion of degree for sets of *multiple* vertices. For example, given an orientation $D(G)$ and a pair of vertices u, v, we can define $d_{12}(u, v)$ to be the number of edges e such that u and v (in some order) are in the first two positions of $D(e)$. So if the oriented edges are $(4, 5, 1), (4, 1, 3), (1, 4, 2)$ (where the vertex set is $[5]$) then $d_{12}(1, 4) = 2$.

More generally, for a p-set of vertices $A = \{v_1, \ldots, v_p\} \subset V$ and a p-set $I \subset [r]$, the *I-degree* of A, denoted by $d_I(A)$, is the number of edges e such that the elements of $D(e)$ in positions labeled by I are $v_1, \ldots, v_p$ (in some order). More formally, $d_I(A)$ is the number of edges e such that if we write $D(e) = (x_1, \ldots, x_r)$ then $\{x_i : i \in I\}$ is exactly the set A.

We mention in passing that there is a variant of this notion where the mutual order of u and v is important. However, this alternative definition turns out to be less interesting