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On totally positive matrices and geometric incidences

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A B S T R A C T

A matrix is called totally positive if every minor of it is positive. Such matrices are well studied and have numerous applications in Mathematics and Computer Science. We study how many times the value of a minor can repeat in a totally positive matrix and show interesting connections with incidence problems in combinatorial geometry. We prove that the maximum possible number of repeated $d \times d$-minors in a $d \times n$ totally-positive matrix is $O(n^{d-\frac{d}{d+1}})$. For the case $d = 2$ we also show that our bound is optimal. We consider some special families of totally positive matrices to show non-trivial lower bounds on the number of repeated minors. In doing so, we arrive at a new interesting problem: How many unit-area and axis-parallel rectangles can be spanned by two points in a set of n points in the plane? This problem seems to be interesting in its own right especially since it seems to have a flavor of additive combinatorics and relates to interesting incidence problems where considering only the topology of the curves involved is not enough. We prove an upper bound of $O(n^{\frac{4}{3}})$ and provide a lower bound of $n^{1+\frac{1}{O(\log \log \sqrt{n})}}$.

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1. Introduction

In incidence geometry a typical problem is to bound the number of incidences between a set of m points and a set of n "well-behaved" surfaces in a given fixed dimensional Euclidean space. The study of geometric incidences has gained momentum in the last few decades. This was proved to be very fruitful. Maybe, the most notable example is the recent dramatic solution of Guth and Katz to the so-called Erdős distinct-distances problem [9].

In this paper we study a combinatorial problem on matrices. Specifically, we would like to bound the number of times a given value can repeat as a minor in a totally positive matrix. Our main contribution is to connect this problem to incidence geometry. Using non-trivial bounds on incidences between points and hyperplanes in $\mathbb{R}^d$ we are able to provide a non-trivial upper-bound on the number of d-by-d minors having a fixed value, say 1.

2. Preliminaries

2.1. Incidences between points and curves

Let L be a set of n distinct lines and let P be a set of m distinct points in the plane. Let $I(P,L)$ denote the number of incidences between points in P and lines in L i.e.,

$$I(P,L) = \left|\left\{(p,l) : p \in P, l \in L, p \in l\right\}\right|$$

and put $I(m,n) = \max_{|P|=m,|L|=n}\{I(P,L)\}$. Erdős provided a construction showing the lower bound $I(m,n) = \Omega((mn)^{2/3} + m + n)$ and also conjectured that this is also the asymptotic upper-bound. His conjecture was settled by Szemerédi and Trotter in their 1983 seminal paper [16]:

Theorem 2.1. *(See [16].)* $I(m,n) = O(m^{\frac{2}{3}}n^{\frac{2}{3}} + m + n)$.

The Szemerédi and Trotter bound can be generalized and be extended in many ways. In this paper we use one of these extensions to the so-called families of pseudolines:

Definition 2.2. A family Γ of n simple Jordan curves in the plane is said to be a family of *pseudolines* if every pair of curves in Γ intersects at most once.

The original proof of Szemerédi and Trotter is rather involved and contains a huge constant hidden in the O-notation. A considerably simpler proof was obtained by Clarkson et al. in [5]. Their paper contains also an extension of the same upper bound for