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Number of cycles in the graph of 312-avoiding permutations

Richard Ehrenborg, Sergey Kitaev [1], Einar Steingrímsson

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ABSTRACT

The graph of overlapping permutations is defined in a way analogous to the De Bruijn graph on strings of symbols. That is, for every permutation $\pi = \pi_1\pi_2\cdots\pi_{n+1}$ there is a directed edge from the standardization of $\pi_1\pi_2\cdots\pi_n$ to the standardization of $\pi_2\pi_3\cdots\pi_{n+1}$. We give a formula for the number of cycles of length d in the subgraph of overlapping 312-avoiding permutations. Using this we also give a refinement of the enumeration of 312-avoiding affine permutations and point out some open problems on this graph, which so far has been little studied.

1. Introduction and preliminaries

One of the classical objects in combinatorics is the *De Bruijn graph*. This is the directed graph on vertex set $\{0, 1, \ldots, q-1\}^n$, the set of all strings of length n over an alphabet of size q, whose directed edges go from each vertex $x_1 \cdots x_n$ to each vertex $x_2 \cdots x_{n+1}$. That is, there is a directed edge from the string $\mathbf{x}$ to the string $\mathbf{y}$ if and only if the last $n-1$ coordinates of $\mathbf{x}$ agree with the first $n-1$ coordinates of $\mathbf{y}$.

The De Bruijn graph has been much studied, especially in connection with combinatorics on words, and one of its well-known properties (see for instance [10, p. 126]) is the

E-mail address: sergey.kitaev@cis.strath.ac.uk (S. Kitaev).
[1] Fax: +44141 5484523.

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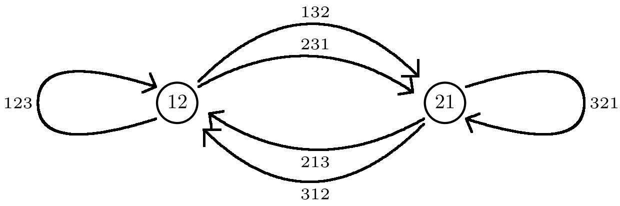

Fig. 1. The graph $G(2)$ of overlapping permutations.

fact that its number of directed cycles of length d, for $d \leq n$, is given by

$$\frac{1}{d}\sum_{e|d}\mu(d/e)q^{e}, \tag{1.1}$$

where the sum is over all divisors e of length d, and where μ denotes the number theoretic *Möbius function*. Recall that $\mu(n)$ is $(-1)^{k}$ if n is a product of k distinct primes and is zero otherwise.

A natural variation on the De Bruijn graphs is obtained by replacing words over an alphabet by permutations of the set of integers $\{1, 2, \dots, n\}$, where the overlapping condition determining directed edges in De Bruijn graphs is replaced by the condition that the head and tail of two permutations have the same *standardization*, that is, that their letters appear in the same order of size. This *graph of overlapping permutations*, denoted $G(n)$, has a directed edge for each permutation $\pi \in \mathfrak{S}_{n+1}$ from the standardization of $\pi_1\pi_2\cdots\pi_n$ to the standardization of $\pi_2\pi_3\cdots\pi_{n+1}$. As an example, there is a directed edge from 2341 to 3412 in $G(4)$ labeled 24513, since the standardizations of 2451 and 4513 are 2341 and 3412, respectively. In fact, between these two vertices there is another directed edge labeled 34512. The simple case of $n = 2$ is illustrated in Fig. 1. Note that, apart from the path and cycle graphs mentioned in Section 3, all graphs in this paper are directed, although we do not explicitly refer to them as directed graphs.

The graph $G(n)$ appeared in [4] in connection with *universal cycles on permutations*. It has also appeared in [7], where it was used as a tool in determining the asymptotic behavior of consecutive pattern avoidance, and in [2], where it is called the *graph of overlapping patterns* (see also [11, Section 5.6]).

What is the number of directed cycles in this graph $G(n)$, that is, the analogue to the question for which Eq. (1.1) is the answer. This is a natural question which does not seem to have been studied so far. We have not been able to solve that problem (and we do not recognize the associated number sequences). We do here, however, solve that problem when the graph is restricted to permutations of length n avoiding the pattern 312, that is, permutations containing no three letters the first of which is the largest and the second of which is the smallest. We prove in Theorem 5.2 that the number of directed cycles of length d in the restriction of the graph is

$$\frac{1}{d}\sum_{e|d}\mu(d/e)\binom{2e}{e}, \tag{1.2}$$