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Categorification and Heisenberg doubles arising from towers of algebras ☆

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ABSTRACT

The Grothendieck groups of the categories of finitely generated modules and finitely generated projective modules over a tower of algebras can be endowed with (co)algebra structures that, in many cases of interest, give rise to a dual pair of Hopf algebras. Moreover, given a dual pair of Hopf algebras, one can construct an algebra called the Heisenberg double, which is a generalization of the classical Heisenberg algebra. The aim of this paper is to study Heisenberg doubles arising from towers of algebras in this manner. First, we develop the basic representation theory of such Heisenberg doubles and show that if induction and restriction satisfy Mackey-like isomorphisms then the Fock space representation of the Heisenberg double has a natural categorification. This unifies the existing categorifications of the polynomial representation of the Weyl algebra and the Fock space representation of the Heisenberg algebra. Second, we develop in detail the theory applied to the tower of 0-Hecke algebras, obtaining new Heisenberg-like algebras that we call *quasi-Heisenberg algebras*. As an application of a generalized Stone–von Neumann Theorem, we give a new proof of the fact that

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the ring of quasisymmetric functions is free over the ring of symmetric functions.

Contents

1. Introduction

The interplay between symmetric groups and the Heisenberg algebra has a rich history, with implications in combinatorics, representation theory, and mathematical physics. A foundational result in this theory is due to Geissinger, who gave a representation theoretic realization of the bialgebra of symmetric functions Sym by considering the Grothendieck groups of representations of all symmetric groups over a field $\Bbbk$ of characteristic zero (see [10]). In particular, he constructed an isomorphism of bialgebras

$$\mathrm{Sym} \cong \bigoplus_{n=0}^{\infty} \mathcal{K}_0\big(\Bbbk[S_n]\text{-mod}\big),$$

where $\mathcal{K}_0(\mathcal{C})$ denotes the Grothendieck group of an abelian category $\mathcal{C}$. Multiplication is described by the induction functor

$$[\mathrm{Ind}] : \mathcal{K}_0\big(\Bbbk[S_n]\text{-mod}\big) \otimes \mathcal{K}_0\big(\Bbbk[S_m]\text{-mod}\big) \to \mathcal{K}_0\big(\Bbbk[S_{n+m}]\text{-mod}\big),$$

while comultiplication is given by restriction. Mackey theory for induction and restriction in symmetric groups implies that the coproduct is an algebra homomorphism. For each S_n-module V, multiplication by the class $[V] \in \mathcal{K}_0(\Bbbk[S_n]\text{-mod})$ defines an endomorphism of $\bigoplus_{n=0}^{\infty} \mathcal{K}_0(\Bbbk[S_n]\text{-mod})$. These endomorphisms, together with their adjoints, define a representation of the Heisenberg algebra on $\bigoplus_{n=0}^{\infty} \mathcal{K}_0(\Bbbk[S_n]\text{-mod})$.

Geissinger's construction was q-deformed by Zelevinsky in [29], who replaced the group algebra of the symmetric group $\Bbbk[S_n]$ by the Hecke algebra $H_n(q)$ at generic q. Again, endomorphisms of the Grothendieck group given by multiplication by classes $[V]$, together with their adjoints, generate a representation of the Heisenberg algebra.