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Distinguishing subgroups of the rationals by their Ramsey properties

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ABSTRACT

A system of linear equations with integer coefficients is *partition regular* over a subset S of the reals if, whenever $S \setminus \{0\}$ is finitely coloured, there is a solution to the system contained in one colour class. It has been known for some time that there is an infinite system of linear equations that is partition regular over $\mathbb{R}$ but not over $\mathbb{Q}$, and it was recently shown (answering a long-standing open question) that one can also distinguish $\mathbb{Q}$ from $\mathbb{Z}$ in this way.

Our aim is to show that the transition from $\mathbb{Z}$ to $\mathbb{Q}$ is not sharp: there is an infinite chain of subgroups of $\mathbb{Q}$, each of which has a system that is partition regular over it but not over its predecessors. We actually prove something stronger: our main result is that if R and S are subrings of $\mathbb{Q}$ with R not contained in S, then there is a system that is partition regular over R but not over S. This implies, for example, that the chain above may be taken to be uncountable.

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1. Introduction

Consider the following system of u linear equations in v unknowns:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,v}x_v & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,v}x_v & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{u,1}x_1 & + & a_{u,2}x_2 & + & \cdots & + & a_{u,v}x_v & = & 0 \end{array}$$

If the coefficients are rational numbers and the set $\mathbb{N}$ of positive integers is finitely coloured, is one guaranteed to be able to find monochromatic $x_1, x_2, \dots, x_v$ solving the given system? That is, is the system of equations *partition regular*? In [8], Rado answered this question, showing that the system is partition regular if and only if the matrix of coefficients

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,v} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{u,1} & a_{u,2} & \cdots & a_{u,v} \end{pmatrix}$$

satisfies the *columns condition*:

Definition 1.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Z}$. Denote column i of A by $\vec{c}_i$. The matrix A satisfies the *columns condition* if there exist $m \in \{1, 2, \dots, v\}$ and a partition $\{I_1, I_2, \dots, I_m\}$ of $\{1, 2, \dots, v\}$ such that

(1) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$;
(2) for each $t \in \{2, 3, \dots, m\}$, if any, $\sum_{i \in I_t} \vec{c}_i$ is a linear combination with coefficients from $\mathbb{Q}$ of $\{\vec{c}_i : i \in \bigcup_{j=1}^{t-1} I_j\}$.

If one considers the same equations over $\mathbb{R}$, an easy compactness argument shows that a finite system of equations is partition regular over the reals if and only if it is partition regular over the integers.

Note that the restriction to integer coefficients might as well be to rational coefficients, as we are always free to multiply each equation by a constant. We remark in passing that if one were to allow coefficients that are not rational, then the situation for finite systems is again understood: in [9], Rado extended his result by showing that if R is any subring of the set $\mathbb{C}$ of complex numbers and the entries of A are from R, then the system of equations is partition regular over R if and only if the matrix A satisfies the columns condition over the field F generated by R (which means that we replace 'linear combination with coefficients from $\mathbb{Q}$' by 'linear combination with coefficients from F').