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Colorful associahedra and cyclohedra

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Dedicated to the memory of Andrei Zelevinsky

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A B S T R A C T

Every n-edge colored n-regular graph $\mathcal{G}$ naturally gives rise to a simple abstract n-polytope, *the colorful polytope of* $\mathcal{G}$, whose 1-skeleton is isomorphic to $\mathcal{G}$. The paper describes colorful polytope versions of the associahedron and cyclohedron. Like their classical counterparts, the colorful associahedron and cyclohedron encode triangulations and flips, but now with the added feature that the diagonals of the triangulations are colored and adjacency of triangulations requires color preserving flips. The colorful associahedron and cyclohedron are derived as colorful polytopes from the edge colored graph whose vertices represent these triangulations and whose colors on edges represent the colors of flipped diagonals.

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1. Introduction

There are interesting connections between abstract polytopes and edge colored regular graphs. Every n-edge colored n-regular graph $\mathcal{G}$ naturally gives rise to a simple abstract n-polytope, called the colorful polytope of $\mathcal{G}$, whose 1-skeleton is isomorphic to $\mathcal{G}$ and whose higher-rank structure is built from $\mathcal{G}$ by following precise instructions encoded in $\mathcal{G}$ (see [1]). In this paper we describe colorful polytope versions of two well-known convex polytopes, the associahedron and the cyclohedron.

The associahedron is a simple convex polytope first described combinatorially by Stasheff [16] in 1963. It is often called the Stasheff polytope. Its combinatorial structure was studied independently by Tamari [17] as a partially ordered set of bracketings of a non-associative product of factors. A realization of this structure as a convex polytope was discovered by Lee [9], as well as by Haiman and Milnor in unpublished work, see also [5] and [12] for excellent references. There are other realizations of the associahedron, for instance, the realization of Shnider & Sternberg [13] using planar binary trees. See also Loday [10] for an algorithm that uses trees to find realizations with integer coordinates. The classical associahedron arises as a special case of the more general construction of secondary polytopes due to Fomin & Zelevinsky [8]. These generalized associahedra also include the cyclohedron, a simple convex polytope first described as a combinatorial object in Bott & Taubes [2] in connection with knot theory, and independently as a geometric polytope by Simion [14].

The ordinary associahedron and cyclohedron can be constructed as convex polytopes that encode triangulations and flips. Their colorful polytope versions, the colorful associahedron and colorful cyclohedron, respectively, are abstract polytopes that similarly encode triangulations and flips, but now with the new feature that the diagonals of the triangulations are colored and adjacency of triangulations requires color preserving flips. The colorful associahedron and colorful cyclohedron are combinatorial coverings of the ordinary associahedron and cyclohedron, respectively.

The paper is organized as follows. We begin in Section 2 with a brief review of basic concepts for graphs and abstract polytopes. Then Sections 3 and 4 investigate the combinatorial structure of the colorful associahedron and establish its covering relationship with the classical associahedron. In Section 5 we determine the automorphism group of the colorful associahedron. Finally, Section 6 describes the structure of the colorful cyclohedron.

The last author wishes to dedicate this paper to the memory of his long-term colleague and friend, Andrei Zelevinsky, who recently passed away.

2. Basic notions

We begin with a brief review of some terminology for graphs and abstract polytopes. See [6,11] for further basic definitions and results.

Let $\mathcal{G}$ be a finite simple graph (without loops or multiple edges), and let $V(\mathcal{G})$ denote its vertex set and $E(\mathcal{G})$ its edge set. An *edge coloring* of $\mathcal{G}$ is an assignment of colors to