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# Perfect matchings in 3-partite 3-uniform hypergraphs

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Let $H$ be a 3-partite 3-uniform hypergraph, i.e. a 3-uniform hypergraph such that every edge intersects every partition class in exactly one vertex, with each partition class of size $n$. We determine a Dirac-type vertex degree threshold for perfect matchings in 3-partite 3-uniform hypergraphs.


## 1. Introduction

A perfect matching in a graph $G$ is a set of vertex-disjoint edges, which covers all vertices of $G$. Tutte [21] gave a characterisation of all graphs that contain a perfect matching. An easy consequence of a celebrated theorem of Dirac [8] is that if $G$ is a graph of even order $n$ and minimum degree $\delta(G) \geq n/2$, then $G$ contains a perfect matching. Thus, it is natural to ask for Dirac-type degree thresholds for perfect matchings in hypergraphs.

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We follow the notation of [4] and denote by $\binom{U}{k}$ the set of all $k$-element subsets of a set $U$. We will often write a $k$-set to mean a $k$-element set. A *$k$-uniform hypergraph*, or *$k$-graph* for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and the edge set $E(H)$ is a set of $k$-subsets of $V(H)$. Often we write $V$ instead of $V(H)$ when it is clear from the context. A matching $M$ in $H$ is a set of vertex-disjoint edges of $H$, and it is *perfect* if $M$ covers all vertices of $H$. Clearly, a perfect matching only exists if $|V|$ is divisible by $k$.

Given a $k$-graph $H$ and an $l$-set $T \in \binom{V}{l}$, let $\deg(T)$ be the number of $(k-l)$-sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in $H$. Let $\delta_l(H)$ be the *minimum $l$-degree of $H$*, that is, $\min \deg(T)$ over all $T \in \binom{V}{l}$. We define $m_l(k,n)$ to be the smallest integer $m$ such that every $k$-graph $H$ of order $n$ satisfying $\delta_l(H) \geq m$ contains a perfect matching. Hence, we always assume that $k|n$ whenever we talk about $m_l(k,n)$. Thus we have $m_1(2,n) = n/2$, by the result of Dirac.

For $k \geq 3$ and $l = k-1$, Rödl, Ruciński and Szemerédi [18] determined the value of $m_{k-1}(k,n)$ exactly, which improved the bound given in [13]. For $k \geq 3$ and $1 \leq l < k$, it is conjectured in [10] that

$$m_l(k,n) \sim \max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-l}\right\}\binom{n}{k-l}. \tag{1}$$

For $k = 3$ and $l = 1$, Hàn, Person and Schacht [10] showed that (1) is true, that is, $m_1(3,n) \sim \frac{5}{9}\binom{n}{2}$ improving on a result of Daykin and Häggkvist [7] for $k = 3$. The exact value was independently determined by Khan [11] and Kühn, Osthus and Treglown [14]. Khan [12] further determined $m_1(4,n)$ exactly. For $k \geq 3$ and $k/2 \leq l < k$, Pikhurko [16] proved that $m_l(k,n) \sim \frac{1}{2}\binom{n}{k-l}$. Recently, exact values of $m_l(k,n)$ for all $k/2 \leq l < k$ were determined by Czygrinow and Kamat [6] and by Treglown and Zhao [19,20]. Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [2] determined the asymptotic value of $m_l(k,n)$ when $k - l \leq 4$. Thus, for $1 \leq l < k/2$, (1) is still open except for a few cases. Partial results were proved by Hàn, Person and Schacht [10] and later improved by the second author and Ruciński [15]. We recommend [17] for a survey of other results on perfect matchings in hypergraphs.

Instead of seeking a perfect matching, Bollobás, Daykin and Erdős [5] considered Dirac-type degree thresholds for a matching of size $m$.

**Theorem 1.1.** *(See Bollobás, Daykin and Erdős [5].) Let $k$ and $m$ be integers with $k \geq 2$. If $H$ is a $k$-graph of order $n \geq 2k^3(m+2)$ and*

$$\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1},$$

*then $H$ contains a matching of size $m$.*