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Nontrivial t-designs over finite fields exist for all t ☆

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ABSTRACT

A t-(n, k, λ) design over $\mathbb{F}_q$ is a collection of k-dimensional subspaces of $\mathbb{F}_q^n$, called blocks, such that each t-dimensional subspace of $\mathbb{F}_q^n$ is contained in exactly λ blocks. Such t-designs over $\mathbb{F}_q$ are the q-analogs of conventional combinatorial designs. Nontrivial t-(n, k, λ) designs over $\mathbb{F}_q$ are currently known to exist only for $t \leqslant 3$. Herein, we prove that simple (meaning, without repeated blocks) nontrivial t-(n, k, λ) designs over $\mathbb{F}_q$ exist for all t and q, provided that $k > 12(t+1)$ and n is sufficiently large. This may be regarded as a q-analog of the celebrated Teirlinck theorem for combinatorial designs.
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1. Introduction

Let X be a set with n elements. A t-(n, k, λ) *combinatorial design* (or t-*design*, in brief) is a collection of k-subsets of X, called blocks, such that each t-subset of X is contained in exactly λ blocks. A t-design is said to be *simple* if there are no repeated blocks — that is, all the k-subsets in the collection are distinct. A *trivial* t-design is the set of all k-subsets of X. The celebrated theorem of Teirlinck [19] establishes the existence of

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nontrivial simple t-designs for all t. A t-(n,k,λ) design with $\lambda = 1$ and $t \geqslant 2$ is said to be a *Steiner system*. In a recent breakthrough work [12], Keevash used the probabilistic method to prove that Steiner systems exist for all t.

It was suggested by Tits [22] in 1957 that combinatorics of sets could be regarded as the limiting case $q \to 1$ of combinatorics of vector spaces over the finite field $\mathbb{F}_q$. Indeed, there is a strong analogy between subsets of a set and subspaces of a vector space, expounded by several authors [6,10,23]. In particular, the notion of t-designs has been extended to vector spaces by Cameron [4,5] and Delsarte [7] in the early 1970s. Specifically, let $\mathbb{F}_q^n$ be a vector space of dimension n over the finite field $\mathbb{F}_q$. Then a t-(n,k,λ) *design over* $\mathbb{F}_q$ is a collection of k-dimensional subspaces of $\mathbb{F}_q^n$ (k-subspaces, for short), called blocks, such that each t-subspace of $\mathbb{F}_q^n$ is contained in exactly λ blocks. Such t-designs over $\mathbb{F}_q$ are the q-analogs of conventional combinatorial designs. As for combinatorial designs, we will say that a t-design over $\mathbb{F}_q$ is *simple* if it does not have repeated blocks, and *trivial* if it is the set of all k-subspaces of $\mathbb{F}_q^n$. A t-(n,k,λ) design over $\mathbb{F}_q$ with $\lambda = 1$ and $t \geqslant 2$ is said to be a *q-Steiner system*. In recent years, there has been increasing interest in designs over fields and, in particular, q-Steiner systems since these were shown in [8,13] to have applications for error-correction in networks, under randomized network coding.

The first examples of simple nontrivial t-designs over $\mathbb{F}_q$ with $t \geqslant 2$ were found by Thomas [20] in 1987. Today, following the work of many authors [1–3,9,15–18,21], numerous such examples are known. In particular, the first nontrivial q-Steiner system, namely a 2-$(13,3,1)$ design over $\mathbb{F}_2$, was recently found in [2]. However, all these examples have $t = 2$ or $t = 3$. In fact, no simple nontrivial t-designs over $\mathbb{F}_q$ are presently known for $t > 3$. Our main result is the following theorem.

Theorem 1. *Simple nontrivial t-(n,k,λ) designs over $\mathbb{F}_q$ exist for all q and t, and all $k > 12(t+1)$ provided that $n \geqslant ckt$ for a large enough absolute constant c. Moreover, these t-(n,k,λ) designs have at most $q^{12(t+1)n}$ blocks.*

This theorem can be regarded as a q-analog of Teirlinck's theorem [19] for combinatorial designs. Our proof of Theorem 1 is based on a new probabilistic procedure introduced by Kuperberg, Lovett, and Peled in [14] to prove the existence of certain regular combinatorial structures. We note that the probabilistic proof technique developed in [14] and invoked in this paper is very different from the probabilistic approach of Keevash [12]. We also note that this proof technique is purely existential: there is no known efficient algorithm which can produce a t-(n,k,λ) design over $\mathbb{F}_q$ for $t > 3$. Hence, we pose the following as an open problem:

$$\text{\textit{Design an efficient algorithm to produce simple nontrivial}}\quad t\text{-}(n,k,\lambda)\ \text{\textit{designs for large}}\ t \tag{$\star$}$$

The rest of this paper is organized as follows. We begin with some preliminary definitions in the next section. We present the Kuperberg–Lovett–Peled (KLP) theorem of

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