J. Combin. Theory Ser. A 127 (2014) 224–242

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

A modular equality for Cameron–Liebler line classes

Alexander L. Gavrilyuk [a,b], Klaus Metsch [c,d]

[a] *Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan*
[b] *N.N. Krasovsky Institute of Mathematics and Mechanics UB RAS, Kovalevskaya str., 16, Ekaterinburg 620990, Russia*
[c] *Justus-Liebig-Universität, Mathematisches Institut, Arndtstraße 2, D-35392 Gießen, Germany*
[d] *Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281-S22, 9000 Gent, Belgium*

A R T I C L E I N F O

Article history:
Received 20 November 2013
Available online 7 July 2014

Dedicated to the memory of Frédéric Vanhove

Keywords:
Cameron–Liebler line class
Tight sets
Hyperbolic quadric
Intriguing set
Completely regular code
Grassmann graph

A B S T R A C T

In this paper we prove that a Cameron–Liebler line class $\mathcal{L}$ in $\mathrm{PG}(3,q)$ with parameter x has the property that $\binom{x}{2}+n(n-x) \equiv 0 \bmod q+1$ for the number n of lines of $\mathcal{L}$ in any plane of $\mathrm{PG}(3,q)$. It follows that the modular equation $\binom{x}{2}+n(n-x) \equiv 0 \bmod q+1$ has an integer solution in n. This result rules out roughly at least one half of all possible parameters x. As an application of our method, we determine the spectrum of parameters of Cameron–Liebler line classes of $\mathrm{PG}(3,5)$. This includes the construction of a Cameron–Liebler line class with parameter 10 in $\mathrm{PG}(3,5)$ and a proof that it is unique up to projectivities and dualities.

E-mail addresses: alexander.gavriliouk@gmail.com (A.L. Gavrilyuk), klaus.metsch@math.uni-giessen.de (K. Metsch).

http://dx.doi.org/10.1016/j.jcta.2014.06.004
0097-3165/

1. Introduction

A *Cameron–Liebler line class* [4,16] of the finite projective space $\mathrm{PG}(3,q)$ is a set of lines that shares a constant number x of lines with every regular spread of $\mathrm{PG}(3,q)$. The number x is called the *parameter* of the Cameron–Liebler line class. These classes appeared in connection with an attempt by Cameron and Liebler [4] to classify collineation groups of $\mathrm{PG}(n,q)$, $n \geq 3$, that have equally many orbits on lines and on points.

An empty set of lines is a Cameron–Liebler line class with parameter $x = 0$. An example with $x = 1$ consists of all lines on a point, a second one of all lines in a plane. If the point is not in the plane, then the union of the previous two examples gives a Cameron–Liebler line class with parameter $x = 2$. There are no other examples with $x \in \{1, 2\}$, see [4]. As the complement of a Cameron–Liebler line class with parameter x is a Cameron–Liebler line class with parameter $q^2 + 1 - x$, there exist also examples with $x = q^2 + 1$, $x = q^2$ and $x = q^2 - 1$. It was conjectured [4] that these are the only Cameron–Liebler line classes. The first counterexamples were constructed by Drudge [8] (in $\mathrm{PG}(3,3)$ with $x = 5$), by Bruen and Drudge [3] (for all odd q, in $\mathrm{PG}(3,q)$ with $x = (q^2 + 1)/2$), the examples are closely related to elliptic quadrics of $\mathrm{PG}(3,q)$, and later by Govaerts and Penttila [10] (in $\mathrm{PG}(3,4)$ with $x = 7$). Despite much effort, see e.g. [18], there is no infinite series known with $x \neq 0, 1, 2, \frac{1}{2}(q^2 + 1), q^2 - 1, q^2, q^2 + 1$.

Connecting Cameron–Liebler line classes to blocking sets, many non-existence results have been proved, see e.g. [6,8,11]. One of the strongest results obtained from the theory of blocking sets is by Govaerts and Storme [11] who showed that $2 < x \leq q$ is impossible when q is a prime. This was improved to the non-existence for $2 < x \leq q$ and all prime powers q [14]. Using a different technique, it was shown in [15] that there exists a constant c such that $x \leq 2$ or $x \geq cq^{4/3}$ for any prime power q. The main result of the present paper is the following modular equation connecting the parameter x with the number of lines of a Cameron–Liebler line class in the planes of $\mathrm{PG}(3,q)$ or dually the number of lines on a point.

Theorem 1.1. *Suppose $\mathcal{L}$ is a Cameron–Liebler line class with parameter x of* $\mathrm{PG}(3,q)$. *Then for every plane and every point of* $\mathrm{PG}(3,q)$

$$\binom{x}{2} + n(n - x) \equiv 0 \mod q + 1 \tag{1}$$

where n is the number of lines of $\mathcal{L}$ in the plane respectively through the point.

It is known, see Result 2.2 in Section 2, that a Cameron–Liebler line class $\mathcal{L}$ with parameter x of $\mathrm{PG}(3,q)$ has the following property. If P is a point and π a plane of $\mathrm{PG}(3,q)$ with $P \in \pi$, then the number of lines of $\mathcal{L}$ through P plus the number of lines of $\mathcal{L}$ in π is congruent to x modulo $q + 1$. Thus, if n is the number of lines of $\mathcal{L}$ in some plane, then every plane has congruent to n modulo $q + 1$ lines of $\mathcal{L}$, and the number of lines of $\mathcal{L}$ through any point is congruent to $x - n$ modulo $q + 1$.