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# A diagrammatic approach to Kronecker squares

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ABSTRACT

In this paper we improve a method of Robinson and Taulbee for computing Kronecker coefficients and show that for any partition $\bar{\nu}$ of $d$ there is a polynomial $k_{\bar{\nu}}$ with rational coefficients in variables $x_C$, where $C$ runs over the set of isomorphism classes of connected skew diagrams of size at most $d$, such that for all partitions $\lambda$ of $n$, the Kronecker coefficient $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ is obtained from $k_{\bar{\nu}}(x_C)$ substituting each $x_C$ by the number of partitions $\alpha$ contained in $\lambda$ such that $\lambda/\alpha$ is in the class $C$. Some results of our method extend to arbitrary Kronecker coefficients. We present two applications. The first is a contribution to the Saxl conjecture, which asserts that if $\rho_k = (k, k-1, \ldots, 2, 1)$ is the staircase partition, then the Kronecker square $\chi^\rho \otimes \chi^\rho$ contains every irreducible character of the symmetric group as a component. Here we prove that for any partition $\bar{\nu}$ of size $d$ there is a piecewise polynomial function $s_{\bar{\nu}}$ in one real variable such that for all $k$, one has $\mathsf{g}(\rho_k, \rho_k, (|\rho_k| - d, \bar{\nu})) = s_{\bar{\nu}}(k)$. The second application is a proof of a new stability property for arbitrary Kronecker coefficients.



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## 1. Introduction

Let $\chi^\lambda$ be the irreducible character of the symmetric group $\mathsf{S}_n$ associated to the partition $\lambda$ of $n$. It is a major open problem in the representation theory of the symmetric group in characteristic 0 to find a combinatorial or geometric description of the multiplicity

$$\mathsf{g}(\lambda,\mu,\nu)=\langle\chi^\lambda\otimes\chi^\mu,\chi^\nu\rangle \tag{1.1}$$

of $\chi^\nu$ in the Kronecker product $\chi^\lambda\otimes\chi^\mu$ of $\chi^\lambda$ and $\chi^\mu$ (here $\langle\cdot,\cdot\rangle$ denotes the scalar product of complex characters). Seventy five years ago Murnaghan [27] published the first paper on the subject. Since then many people have searched out satisfactory ways for computing the Kronecker coefficients $\mathsf{g}(\lambda,\mu,\nu)$. Still, very little is known about the general problem.

Among the things known, there is a method for computing arbitrary Kronecker coefficients. It was introduced by Robinson and Taulbee in [37] (see also [36, §3.4]) and reworked by Littlewood in [21]. In [18, §2.9] we can find the original method of Robinson and Taulbee, another variation of it and some applications. We will refer to this method and to any of its variations as the RT method. Its main ingredients are the Jacobi–Trudi determinant, Frobenius reciprocity and the Littlewood–Richardson rule. Some of its applications can be found in [1,39,43,47]. Another variation of the RT method appears in [17, §6]. Some applications of this variation are given in [3,4,35].

The version of the RT method from [18, p. 98] suggests how to systematize it by means of the so-called Littlewood–Richardson multitableaux (or simply LR multitableaux), see [13,43]. This technique, as it is already apparent from [18], is not only useful for computations: it also lead in [44] to a combinatorial proof of a stability property for Kronecker coefficients observed by Murnaghan in [27] and to the determination of a lower bound for stability. Other approaches to stability have been developed in [9,10,22,42]. A dual approach of LR multitableaux was used in [45] to study minimal components, in the dominance order of partitions, of Kronecker products. LR multitableaux were also used in [2] to construct a one-to-one correspondence between the set 3-dimensional matrices with integer entries and given 1-marginals and the set of certain triples of tableaux. This correspondence generalize the RSK correspondence and was used to describe combinatorially some Kronecker coefficients.

In [43] we gave graphical formulas for the coefficients $\mathsf{g}(\lambda,\lambda,\nu)$ of Kronecker squares for all partitions $\nu=(n-d,\bar\nu)$ of depth $d\le 3$. These computations were extended in [1] to all partitions $\nu$ of depth 4. We include them in Section 7 for completeness. Some of them had appeared before in an algebraic but equivalent form in [16,39,47]; some have already been applied in [6–8,34]; others may be suitable for future applications.

In this paper we prove that the formulas obtained in [43] and [1] are part of a general phenomenon (Theorem 7.2). Namely, for each partition $\bar\nu$ of size $d$, there is a polynomial $k_{\bar\nu}(x_C)$ with rational coefficients in variables $x_C$, where $C$ runs over the set of isomor-