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Supersolvable restrictions of reflection arrangements

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathcal{A} = (\mathcal{A}, V)$ be a complex hyperplane arrangement and let $L(\mathcal{A})$ denote its intersection lattice. The arrangement $\mathcal{A}$ is called supersolvable, provided its lattice $L(\mathcal{A})$ is supersolvable. For X in $L(\mathcal{A})$, it is known that the restriction $\mathcal{A}^X$ is supersolvable provided $\mathcal{A}$ is.

Suppose that W is a finite, unitary reflection group acting on the complex vector space V. Let $\mathcal{A} = (\mathcal{A}(W), V)$ be its associated hyperplane arrangement. Recently, the last two authors classified all supersolvable reflection arrangements. Extending this work, the aim of this note is to determine all supersolvable restrictions of reflection arrangements. It turns out that apart from the obvious restrictions of supersolvable reflection arrangements there are only a few additional instances. Moreover, in a recent paper we classified all inductively free restrictions $\mathcal{A}(W)^X$ of reflection arrangements $\mathcal{A}(W)$. Since every supersolvable arrangement is inductively free, the supersolvable restrictions $\mathcal{A}(W)^X$ of reflection arrangements $\mathcal{A}(W)$ form a natural subclass of the class of inductively free restrictions $\mathcal{A}(W)^X$.

Finally, we characterize the irreducible supersolvable restrictions of reflection arrangements by the presence of modular elements of dimension 1 in their intersection lattice. This in turn leads to the surprising fact that reflection arrangements

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as well as their restrictions are supersolvable if and only if they are strictly linearly fibered.

1. Introduction

Let $\mathcal{A} = (\mathcal{A}, V)$ be a complex hyperplane arrangement and let $L(\mathcal{A})$ denote its intersection lattice. We say that $\mathcal{A}$ is *supersolvable*, provided $L(\mathcal{A})$ is supersolvable, see Definition 2.3. Thanks to [10, Prop. 3.2] (see Corollary 2.7), for X in $L(\mathcal{A})$, the restriction $\mathcal{A}^X$ of a supersolvable arrangement $\mathcal{A}$ is itself again supersolvable.

Now suppose that W is a finite, unitary reflection group acting on the complex vector space V. Let $\mathcal{A} = (\mathcal{A}(W), V)$ be the associated hyperplane arrangement of W. In [4, Thm. 1.2], we classified all supersolvable reflection arrangements. Extending that earlier work, the aim of this note is to classify all supersolvable restrictions $\mathcal{A}^X$ for $\mathcal{A}$ a reflection arrangement. Since supersolvability is a rather strong condition, not unexpectedly, there are only very few additional instances apart from the obvious restrictions of supersolvable reflection arrangements.

Moreover, similar to the case of supersolvable reflection arrangements, we are able to characterize the irreducible arrangements in this class merely by the presence of modular elements of dimension 1 in their intersection lattice (see Theorem 1.5). This in turn leads to the unexpected, remarkable fact that reflection arrangements as well as their restrictions are supersolvable if and only if they are strictly linearly fibered (see Corollary 1.7).

The classification of the irreducible, finite, complex reflection groups W due to Shephard and Todd [9] states that each such group belongs to one of two types. Namely, either W belongs to the infinite three-parameter family $G(r, p, \ell)$ of monomial groups, or else is one of an additional 34 exceptional groups, simply named G_4 up to G_{37}. As a result, proofs of properties of W and its arrangement $\mathcal{A}(W)$ frequently also do come in two flavors: conceptional, uniform arguments for the infinite families on the one hand, and ad hoc and mere computational techniques for the exceptional instances, on the other, e.g. see [7, §6, App. B, App. C] and [6]. This dichotomy also prevails the statements and proofs of this paper.

First we recall the main result from [4, Thm. 1.2]:

Theorem 1.1. *For W a finite complex reflection group, $\mathcal{A}(W)$ is supersolvable if and only if any irreducible factor of W is of rank at most 2, or is isomorphic either to a Coxeter group of type A_ℓ or B_ℓ for $\ell \geq 3$, or to a monomial group $G(r, p, \ell)$ for $r, \ell \geq 3$ and $p \neq r$.*

It is easy to see that any central arrangement of rank at most 2 is supersolvable, cf. [4, Rem. 2.3]. Thus we focus in the sequel on restrictions $\mathcal{A}^X$ with $\dim X \geq 3$.