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A one-parameter refinement of the Razumov–Stroganov correspondence

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ABSTRACT

We introduce and prove a one-parameter refinement of the Razumov–Stroganov correspondence. This is achieved for fully-packed loop configurations (FPL) on domains which generalise the square domain, and which are endowed with the gyration operation. We consider one given side of the domain, and FPLs such that the only straight-line tile on this side is black. We show that the enumeration vector associated with such FPLs, weighted according to the position of the straight line and refined according to the link pattern for the black boundary points, is the ground state of the *scattering matrix*, an integrable one-parameter deformation of the $O(1)$ Dense Loop Model Hamiltonian. We show how the original Razumov–Stroganov correspondence, and a conjecture formulated by Di Francesco in 2004, follows from our results.

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1. Introduction

The *Razumov–Stroganov correspondence* [13,3] relates some fine statistical properties of two distinct integrable systems in Statistical Mechanics [1]: on one side the *6-Vertex Model* on portions of the square lattice, with domain-wall boundary conditions, and on the other side the $O(1)$ *Dense Loop Model* (DLM) with cyclic boundary conditions.

On an $n \times n$ square domain, the configurations of the first model have several easy reformulations, in terms of *fully-packed loops* (FPL), *Alternating Sign Matrices*, or a family of monotone arrays called *Gog triangles* [2]. In the FPL incarnation, to each configuration ϕ is naturally associated a non-crossing pairing $\pi = \pi(\phi)$ of a set of cyclically-ordered points (called a *link pattern*). We call $\Psi_{\mathrm{FPL}}(\pi)$ the corresponding enumerations, i.e. the number of ϕ's such that $\pi(\phi) = \pi$, and $Z_{\mathrm{FPL}} = \sum_\pi \Psi_{\mathrm{FPL}}(\pi)$ their total number. An explicit formula is known for this quantity [17,11],

$$Z_{\mathrm{FPL}}(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

For the second model, we have reformulations in terms of the *integrable XXZ quantum spin chain* at $\Delta = -\frac{1}{2}$, and the *Potts Model* at the percolation point, $Q = 1$. The model is realised on a semi-infinite cylinder, and is naturally analysed through transfer-matrix techniques. In the DLM incarnation, the transfer matrix $T_{\pi\pi'}$ acts on a space whose states are naturally labeled by link patterns. The matrix T encodes the transition rates of a Markov Chain on this space, and there is a unique steady-state distribution $\Psi_{O(1)}(\pi)$, called the *ground state*, and corresponding to the Perron–Frobenius right eigenvector of T. Integrability shows that T commutes with a simpler operator, the *Hamiltonian*, H_0, so that $\Psi_{O(1)}(\pi)$ is also a right eigenvector of H_0. As the corresponding left eigenvector is the uniform vector, with all entries equal to 1, the natural norm of $\Psi_{O(1)}(\pi)$ is given by the sum of the entries, $Z_{O(1)} = \sum_\pi \Psi_{O(1)}(\pi)$.

The Razumov–Stroganov correspondence states that, under the normalisation for $\Psi_{O(1)}$ that sets $Z_{O(1)} = Z_{\mathrm{FPL}}$, we have $\Psi_{\mathrm{FPL}}(\pi) = \Psi_{O(1)}(\pi)$ for all link patterns π. This fact was conjectured in [13], and proven by the authors in [3].

A great effort has been devoted to the study of the properties of the ground state of the $O(1)$ Dense Loop Model. Building on the integrable structure of the DLM, some deep connections with the representation theory of $U_q(\widehat{sl_2})$, or of affine Hecke algebras, have been elucidated, and even connections with algebraic geometry have emerged (see [19] for a review).

The power of integrability manifests itself when the original loop model is deformed by introducing the so-called *spectral parameters* $\vec{z} = \{z_i\}$ (the uniform counting corresponds to the choice $z_i = 1$ for all i). The components of the ground state, $\Psi_{O(1)}(\pi)$, that, in the uniform model and after normalisation, are all integers, are deformed into polynomials in these parameters. Besides the emergence of the connections mentioned above, this procedure has more concretely enabled the derivation of closed formulas for