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Bounds on the number of small Latin subsquares

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ABSTRACT

Let $\zeta(n,m)$ be the largest number of order m subsquares achieved by any Latin square of order n. We show that $\zeta(n,m)=\Theta(n^3)$ if $m\in\{2,3,5\}$ and $\zeta(n,m)=\Theta(n^4)$ if $m\in\{4,6,9,10\}$. In particular, $\frac{1}{8}n^3+O(n^2)\leqslant\zeta(n,2)\leqslant\frac{1}{4}n^3+O(n^2)$ and $\frac{1}{27}n^3+O(n^{5/2})\leqslant\zeta(n,3)\leqslant\frac{1}{18}n^3+O(n^2)$ for all n. We find an explicit bound on $\zeta(n,2^d)$ of the form $\Theta(n^{d+2})$ and which is achieved only by the elementary abelian 2-groups.

For a fixed Latin square L let $\zeta^*(n,L)$ be the largest number of subsquares isotopic to L achieved by any Latin square of order n. When L is a cyclic Latin square we show that $\zeta^*(n,L)=\Theta(n^3)$. For a large class of Latin squares L we show that $\zeta^*(n,L)=O(n^3)$. For any Latin square L we give an ϵ in the interval $(0,1)$ such that $\zeta^*(n,L)\geqslant\Omega(n^{2+\epsilon})$. We believe that this bound is achieved for certain squares L.

1. Introduction

A *Latin square* of order n is an $n\times n$ array of n symbols in which each symbol occurs exactly once in each row and in each column. Any submatrix that is itself a Latin square is called a *subsquare*. An *entry* in a Latin square is a triple specifying a row and column and the symbol in that position. Two Latin squares of the same order and having the

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same symbol set are *isotopic* if one can be obtained from the other by permuting rows, columns and symbols.

Define $\zeta(n, m)$ to be the largest number of order m subsquares achieved by any Latin square of order n. We say that a Latin square of order n achieves $\zeta(n, m)$ if it has $\zeta(n, m)$ subsquares of order m. In this paper we consider the basic question of how large $\zeta(n, m)$ can be. This question was first considered for *intercalates*, that is, subsquares of order 2. Heinrich and Wallis [9] proved that

Theorem 1. *$\zeta(n, 2) \leqslant \frac{1}{4}n^2(n-1)$ for all n, and a Latin square achieves $\zeta(n, 2)$ if and only if it is isotopic to an elementary abelian 2-group.*

Statements equivalent to Theorem 1, but not phrased in terms of Latin squares, had been earlier shown in [5,15], and possibly other places. In Theorem 12, we prove a generalisation of Theorem 1 to subsquares of order any power of 2. We also prove, in Theorem 14, a lower bound on $\zeta(n, 2)$ that is within a factor of $2 + O(1/n)$ of the upper bound.

The next case to be considered was order 3 subsquares. Van Rees [14] showed:

Theorem 2. *$\zeta(n, 3) \leqslant \frac{1}{18}n^2(n-1)$.*

Equality is achieved in Theorem 2 by the elementary abelian 3-groups, but also by many other examples. Nevertheless, van Rees conjectured that equality could only be achieved for orders that are powers of 3. This conjecture remains open, but see [10] for some modest progress.

Upper bounds for $\zeta(n, m)$ for general n and m have recently been given. In [3] it was shown that $\zeta(n, m) \leqslant O(n^{\sqrt{2m}+2})$. This bound was then improved in [4], by showing that $\zeta(n, m) \leqslant n^{\lceil \log_2 m \rceil + 2}$. However these papers were primarily focused on asymptotic methods for large m and in this paper we concentrate on fixed m and general n. We use standard asymptotic O, Θ and Ω notation, and always the asymptotics are for $n \to \infty$ with all other parameters fixed.

As well as considering the total number of order m subsquares we also consider the question of how many copies of a particular subsquare M we might find. Define $\zeta^*(n, M)$ to be the maximum, over all Latin squares of order n, of the number of subsquares that are isotopic to M. In the case when M is the Cayley table of a cyclic group, we show that $\zeta^*(n, M) = \Theta(n^3)$. There exists $\epsilon > 0$ that depends on m but not on M, such that $\zeta^*(n, M) \geqslant n^{2+\epsilon}$. We show that $\zeta^*(n, M) \leqslant O(n^3)$ for many choices of M and conjecture that $\zeta^*(n, M) \leqslant o(n^3)$ for asymptotically almost all choices of M.

Cayley tables of groups are a fertile source of examples for our study, since they are often rich in subsquares. Also, any upper bound we prove for the number of subsquares in Latin squares of order n implies an upper bound on the number of subgroups in groups of order n. We use $\mathbb{Z}_n$ and D_n, respectively, to denote the cyclic group and dihedral group of order n. For any group G, we interpret $\zeta^*(n, G)$ to mean $\zeta^*(n, M)$ where M is a Cayley table for G.