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Half-arc-transitive group actions with a small number of alternets

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ABSTRACT

A graph X is said to be *G-half-arc-transitive* if $G \leqslant \mathrm{Aut}(X)$ acts transitively on the set of vertices of X and on the set of edges of X but does not act transitively on the set of arcs of X. Such graphs can be studied via corresponding alternets, that is, equivalence classes of the so-called reachability relation, first introduced by Cameron, Praeger and Wormald (1993) in [5]. If the vertex sets of two adjacent alternets either coincide or have half of their vertices in common the graph is said to be *tightly attached*. In this paper graphs admitting a half-arc-transitive group action with at most five alternets are considered. In particular, it is shown that if the number of alternets is at most three, then the graph is necessarily tightly attached, but there exist graphs with four and graphs with five alternets which are not tightly attached. The exceptional graphs all admit a partition giving the rose window graph $R_6(5,4)$ on 12 vertices as a quotient graph in case of four alternets, and a particular graph on 20 vertices in the case of five alternets.

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1. Introduction

Throughout this paper graphs are finite, simple, connected and undirected (but with an implicit orientation of the edges when appropriate), and groups are finite, unless specified otherwise. Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the sets of its vertices, edges, arcs and the automorphism group of X, respectively. A subgroup $G \leqslant \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if it acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. Moreover, a subgroup $G \leqslant \mathrm{Aut}(X)$ is said to be *half-arc-transitive* if it is vertex- and edge- but not arc-transitive. A graph X is said to be *G-vertex-transitive*, *G-edge-transitive*, *G-arc-transitive* and *G-half-arc-transitive* if the subgroup $G \leqslant \mathrm{Aut}(X)$ is vertex-transitive, edge-transitive, arc-transitive, and half-arc-transitive, respectively. A G-half-arc-transitive graph is also called *G-half-transitive* or *G-semi-transitive* in the literature [30,33]. When $G = \mathrm{Aut}(X)$ the symbol G will be omitted.

Tutte in [29] observed that the valency of a half-arc-transitive graph is even. In [4] Bouwer gave a construction of a half-arc-transitive graph of valency $2k$ for any integer $k \geqslant 2$. (Note that the smallest half-arc-transitive graph is the Doyle–Holt graph [1,6,10], a quartic graph of order 27.)

Half-arc-transitive graphs, quartic half-arc-transitive graphs in particular, and graphs admitting half-arc-transitive group actions in general have recently become an active topic of research. Various constructions of such graphs together with their structural properties are known, and a classification of certain restricted families of half-arc-transitive graphs has also been obtained, see [2,7,13–16,19,20,23,25–28,30–33,35]. There are several approaches used in that respect, ranging from more algebraic in nature, such as the investigation of (im)primitivity (of half-arc-transitive group actions on graphs), to those which are more geometric and/or combinatorial in nature, such as for example the reachability relation approach, explained below.

Let X be a graph of valency $2k$, $k \geqslant 2$, admitting a half-arc-transitive action of a subgroup $G \leqslant \mathrm{Aut}(X)$. Further, let $D_G(X)$ be one of the two oppositely oriented digraphs associated with X with respect to the action of G. (Choose the orientation on an arbitrary edge, the action of G then defines the orientations of the remaining edges. These two digraphs correspond to two paired orbital digraphs associated with G.) We shall say that two directed edges are "related" if they have the same initial vertex, or the same terminal vertex. This gives rise to an equivalence relation, called the *reachability relation* (see [5,17] where this concept is considered in a larger context of infinite graphs). The subgraphs consisting of equivalence classes of directed edges of the reachability relation are called *G-alternating cycles* when X is of valency 4 (see [19,22,24]), and *G-alternets* in the general case [33]. Clearly, the alternets are blocks of imprimitivity for the action of $\mathrm{Aut}(D_G(X))$ on the edges of $D_G(X)$. (For a transitive group G acting transitively on the set Ω, a partition $\mathcal{B}$ of Ω is said to be *G-invariant* if the elements of G permute, setwise, the parts of $\mathcal{B}$, called *blocks of imprimitivity*.)