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A refinement of Wilf-equivalence for patterns of length 4

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ABSTRACT

In their paper [6], Dokos et al. conjecture that the major index statistic is equidistributed among 1423-avoiding, 2413-avoiding, and 3214-avoiding permutations. In this paper we confirm this conjecture by constructing two major index preserving bijections, $\Theta : S_n(1423) \to S_n(2413)$ and $\Omega : S_n(3214) \to S_n(2413)$. In fact, we show that Θ (respectively, Ω) preserves numerous other statistics including the descent set, right-to-left maxima (respectively, left-to-right minima), and a statistic we call steps. Additionally, Θ (respectively, Ω) fixes all permutations avoiding both 1423 and 2413 (respectively, 3214 and 2413).

1. Introduction

We write a permutation $\pi \in S_n$ as $\pi_1 \ldots \pi_n$ where $|\pi| = n$ is called the length of π. For any sequence of distinct integers $a_1 \ldots a_n$ there is a unique permutation $\pi \in S_n$ with the defining property that $a_i < a_j$ if and only if $\pi_i < \pi_j$, provided $i \neq j$. We say that $a_1 \ldots a_n$ is *order isomorphic* to π and write $\pi = \mathrm{std}(a_1 \ldots a_n)$. For example, $1342 = \mathrm{std}(2693)$. Moreover, we say that π *avoids* $\tau \in S_m$ if no subsequence of π is order isomorphic to τ. We denote by $S_n(\tau)$ the set of all $\pi \in S_n$ that avoid τ. In this context we usually refer to τ as a *pattern*. For example, 5734612 avoids 1423 but does not avoid 2413.

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If two patterns $\sigma, \tau \in S_m$ are such that $|S_n(\tau)| = |S_n(\sigma)|$ for all n, then we say that σ is *Wilf-equivalent* to τ, and write $\sigma \sim \tau$. For example, it is well known that S_3 has only one Wilf-equivalence class, while S_4 partitions into 3 classes [3].

A refinement of Wilf-equivalence involves the idea of a *permutation statistic*, which is defined to be a function $f : S_n \to T$, where T is any fixed set. We say σ and τ are f-Wilf-equivalent if, for all n, there is some bijection $\Theta : S_n(\sigma) \to S_n(\tau)$ such that $f(\pi) = f(\Theta(\pi))$. In other words, the f statistic is equally distributed on the sets $S_n(\sigma)$ and $S_n(\tau)$. In terms of the bijection Θ, we say that Θ *preserves* f. This refinement has been heavily studied for patterns of length 3, for example see [1,2,5,7,8]. A particularly nice (and nearly exhaustive) classification of Wilf-equivalent patterns of length 3 and permutation statistics is given by Claesson and Kitaev in [4]. On the other hand, little is known about permutation statistics and patterns of length 4 or greater. To state a recent conjecture by Dokos et al. [6, Conjecture 2.8] with regards to permutation statistics and patterns of length 4 we need the following definition. First, we say that i is a *descent* in π if $\pi_i > \pi_{i+1}$ and we denote the set of all descents in π by $\operatorname{Des}\pi$. The *major index* is then defined to be $\operatorname{maj}(\pi) = \sum_{i \in \operatorname{Des}\pi} i$. With these definitions, Dokos et al. state the following conjecture:

Conjecture 1. *The major index is equally distributed on the sets $S_n(2413)$, $S_n(1423)$, and $S_n(2314)$.*

In a private communication, S. Elizalde conjectured further that the descent sets are equally distributed on the sets $S_n(1423)$ and $S_n(2413)$. Additionally, B. Sagan conjectured that the positions of the n and the $n-1$ are equally distributed on these two sets as well. In this paper, we confirm Conjecture 1, as well as these stronger claims, by constructing explicit bijections $\Theta : S_n(1423) \to S_n(2413)$ and $\Omega : S_n(2314) \to S_n(2413)$ that preserve the descent set and, in the case of Θ, preserve the position of the n and the $n-1$. In fact, we show that these maps simultaneously preserve several other permutation statistics. To describe these permutation statistics we require a few definitions.

First, a *right-to-left maximum* in π is an index i such that $\pi_i > \pi_j$ provided $i < j$. We denote by $RL(\pi)$ the set of all right-to-left maxima in π. Denoting $RL(\pi) = \{i_1 < \cdots < i_s\}$ we observe that $\pi_{i_1} = n$, $i_s = n$, $\pi_{i_1} > \cdots > \pi_{i_s}$, and $\pi_{i_j} > \pi_k$, provided $i_j < k < i_{j+1}$.

Similarly, we define a *left-to-right minimum* in π to be an index i such that $\pi_j > \pi_i$ provided $j < i$. We denote by $LR(\pi)$ the set of all left-to-right minima in π.

The last permutation statistics we will need to consider are called *steps*, which are defined as follows. A step in π is an index i such that $\pi_i - 1 = \pi_{i+1}$. The set of all steps in π is denoted by $\operatorname{Step}\pi$.

In the next section we prove the existence of a bijection $\Theta : S_n(1423) \to S_n(2413)$ that preserves descents, right-to-left maxima, steps, and the position of the n and $n-1$. As a corollary, we show the existence of an Ω that preserves descents, left-to-right minima, steps, and the positions of the 1 and the 2.