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An improved bound on the existence of Cameron–Liebler line classes

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ABSTRACT

In this paper it is shown that there exists a constant $c > 0$ such that there do not exist Cameron–Liebler line classes in $\mathrm{PG}(3, q)$ with parameter x satisfying $2 < x < cq^{4/3}$. This improves all known bounds except for small values of q.

1. Introduction

Cameron–Liebler line classes were introduced in [3]. They are sets of lines in a 3-dimensional projective space $\mathrm{PG}(3, q)$ with the property that they share a constant number x of lines with every regular spread. Such a set necessarily has $x(q^2 + q + 1)$ lines and x is called its parameter. Then $0 \leqslant x \leqslant q^2 + 1$ with equality on either side, if the line set is trivial, that is the empty set or the set of all lines. For other characterizations as well as for the motivation to study these sets, we refer to the original paper of Cameron and Liebler and to the papers mentioned in the reference list.

In general the complement of a Cameron–Liebler line class with parameter x is a Cameron–Liebler line class with parameter $q^2 + 1 - x$, so it is sufficient to study parameters $x \leqslant \frac{1}{2}(q^2 + 1)$. All lines on a point, or dually, all lines in a plane provide a Cameron–Liebler line class with parameter $x = 1$. Given a non-incident point–plane pair, the set consisting of all lines that contain the point or are contained in the plane is an example with $x = 2$. It is known that every Cameron–Liebler line class with parameter $x \in \{1, 2\}$ is as just described. The only other known infinite series (apart from taking complements) was described by Bruen and Drudge [2] and has parameter $x = \frac{1}{2}(q^2 + 1)$. It is possible that there exist other infinite families but no general construction is known. One paper supporting a conjecture for more infinite families is by Rodgers [11], who constructed examples with parameter

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$x = \frac{1}{2}(q^2 - 1)$ for all prime powers $q \leqslant 200$ with $q \equiv 5 \mod 12$ or $q \equiv 9 \mod 12$. There exist also examples with $x = \frac{1}{3}(q+1)^2$ when $q \equiv 2 \mod 3$ for certain small values of q.

On the other side there has been much effort to show that certain values do not occur as parameters of Cameron–Liebler line classes. Penttila [10] proved the non-existence for $x = 3$ for all q, and the non-existence for $x = 4$ when $q \geqslant 5$. It was a big step to connect Cameron–Liebler line classes to the theory of blocking sets. This was done by Drudge who proved non-existence for $2 < x < \sqrt{q}$, or more general $2 < x < b - 1 - q$, where $b = b(q)$ is the size of the smallest non-trivial blocking set in $\mathrm{PG}(2,q)$. The case $(x,q) = (4,3)$ was excluded by Drudge [5]. The cases $(x,q) = (4,4)$ and $(x,q) = (5,4)$ were excluded by Govaerts and Penttila [7] and in the same article an example with $(x,q) = (7,4)$ was constructed, in fact this was the first known example with $2 < x < q^2 - 1$ and even q. More results obtained by relating Cameron–Liebler line classes to blocking sets are proven by Govaerts and Storme [8], one of their results is that $2 < x \leqslant q$ is impossible if q is a prime. After De Beule, Hallez and Storme [4] have shown the non-existence for $2 < x \leqslant q/2$ for all prime powers q, the non-existence for $2 < x \leqslant q$ for all prime powers q was proved in [9]. Finally Beukemann [1] showed that $x \neq q + 1$ for primes $q > 3$.

It seems unlikely that properties of blocking sets provide enough information to make substantial progress on improving the above results. Here we show that a quite easy argument of combinatorial and geometric nature gives a result that is much stronger than the above mentioned bounds on x.

Theorem 1.1. *If $\mathcal{L}$ is a Cameron–Liebler line class with parameter x of* $\mathrm{PG}(3,q)$, *then $x \leqslant 2$ or $x > q\sqrt[3]{q/2} - \frac{2}{3}q$.*

It might be true that there is a constant c such that $2 < x < cq^2$ is impossible for the parameter x of a Cameron–Liebler line class. In order to prove such a result or to characterize Cameron–Liebler line classes with sufficiently small x, the second theorem of this paper might help.

Theorem 1.2. *Suppose that $\mathcal{L}$ is a Cameron–Liebler line class with parameter x of* $\mathrm{PG}(3,q)$ *and that $x \leqslant (5 - 2\sqrt{6})(q+1)^2$. If there is an incident point–plane pair (P, π) such that the $q+1$ lines that are incident with P and π belong to $\mathcal{L}$, then all lines incident with P or all lines incident with π belong to $\mathcal{L}$.*

Remark. Under the Klein-correspondence Cameron–Liebler line classes correspond to tight sets in the hyperbolic quadric $Q^+(5,q)$, see the next section. Theorem 1.2 for tight sets reads as follows. Suppose that M is a tight set of $Q^+(5,q)$ with $|M| = x(q^2+q+1)$ and $x \leqslant (5 - 2\sqrt{6})(q+1)^2$. Then every line contained in M lies in a plane that is contained in M.

2. The proof

Under the Klein-correspondence Cameron–Liebler line classes translate to tight sets of $Q^+(5,q)$. In order to define tight sets directly in $Q^+(5,q)$, we denote the ambient space of $Q^+(5,q)$ by $\mathrm{PG}(5,q)$ and the related polarity by $\perp$. A tight set of $Q^+(5,q)$ can then be defined as a set of points that meets every subspace $l^\perp$, l a line of $\mathrm{PG}(5,q)$ missing $Q^+(5,q)$, in the same number x of points. An easy counting argument shows that this implies that $|M| = x(q^2+q+1)$. Note that $l^\perp \cap Q^+(5,q)$ for a line l as above is an elliptic quadric $Q^-(3,q)$, which corresponds under the Klein-correspondence to a regular spread of $\mathrm{PG}(3,q)$.

The proof of the two main theorems will be carried out for tight sets of $Q^+(5,q)$. Throughout this section M denotes a tight set of $Q^+(5,q)$ with $|M| = x(q^2+q+1)$. We will make use of the following results.

Result 2.1. *(See [10, Theorem 1(viii)].) If l is a line of the ambient space* $\mathrm{PG}(5,q)$ *of $Q^+(5,q)$, then $|l^\perp \cap M| = q|l \cap M| + x$. In particular, if l is a line of the quadric $Q^+(5,q)$ and π and τ are the two planes of the quadric $Q^+(5,q)$ on l, then $|\pi \cap M| + |\tau \cap M| = (q+1)|l \cap M| + x$.*

Result 2.2. *(See [6].) Let P be a point of the quadric $Q^+(5,q)$, let l_i, $1 \leqslant i \leqslant (q+1)^2$, be the lines on P of $Q^+(5,q)$, and put $t_i := |(l_i \setminus \{P\}) \cap M|$.*