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Bruhat order on plane posets and applications

Loïc Foissy

Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, Université du Littoral Côte d'Opale, Centre Universitaire de la Mi-Voix, 50, rue Ferdinand Buisson, CS 80699, 62228 Calais Cedex, France

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ABSTRACT

A plane poset is a finite set with two partial orders, satisfying a certain incompatibility condition. The set $\mathcal{PP}$ of isoclasses of plane posets owns two products, and an infinitesimal unital bialgebra structure is defined on the vector space $\mathcal{H}_{\mathcal{PP}}$ generated by $\mathcal{PP}$, using the notion of biideals of plane posets. We here define a partial order on $\mathcal{PP}$, making it isomorphic to the set of partitions with the weak Bruhat order. We prove that this order is compatible with both products of $\mathcal{PP}$; moreover, it encodes a nondegenerate Hopf pairing on the infinitesimal unital bialgebra $\mathcal{H}_{\mathcal{PP}}$.

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E-mail address: foissy@lmpa.univ-littoral.fr.

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1. Introduction

In [16], Malvenuto and Reutenauer introduced the notion of *double poset* is a finite set with two partial orders. The set of (isoclasses of) double posets owns several algebraic structures, as:

- a product called *composition*; it corresponds, roughly speaking, to the concatenation of Hasse graphs;
- a coproduct, defined with the notion of *ideals* for the first partial order. One obtains in this way the Malvenuto–Reutenauer Hopf algebra of double posets [16];
- a pairing defined with the help of Zelevinsky *pictures* [9,11,21,22]. It is shown in [16] that this pairing is Hopf; consequently, the Hopf algebra of double posets is free, cofree and self-dual.

This Hopf algebra also contains many interesting subobjects, as, for example, the Hopf algebra of *special posets*, that is to say double posets such that the second partial order is total, a notion related to Stanley's (P, ω)-posets [18,14], the Hopf algebra of plane posets [8,7], that is to say double posets such that the two partial orders satisfy an incompatibility condition (see Definition 1 below), or the noncommutative Hopf algebra of plane trees [4,5,10], also known as the noncommutative Connes–Kreimer Hopf algebra. In particular, the Hopf subalgebra of plane posets turns out to be isomorphic to the Hopf algebra of permutations introduced by Malvenuto and Reutenauer in [15], also known as the Hopf algebra of free quasi-symmetric functions [2,1]. An explicit isomorphism can be defined with the help of a bijection Ψ_n between the set of plane posets on n vertices and the symmetric group on n letters, recalled here in Theorem 3. This isomorphism and its applications are studied in [8].

We proceed here with the algebraic study of the links between permutations and plane posets. As the symmetric group $\mathfrak{S}_n$ is partially ordered by the weak Bruhat order, via the bijection Ψ_n the set of plane posets is also partially ordered. This order has a nice combinatorial description, see Definition 8. It admits a decreasing bijection ι, given by the exchange of the two partial orders defining plane posets; on the permutation side, this bijection consists of reversing the words representing the permutations. For example, let us give the Hasse graph of this partial order restricted to plane posets of degree 3, and the Hasse graph of the weak Bruhat order on $\mathfrak{S}_3$: