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A polynomial embedding of pairs of orthogonal partial Latin squares

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ABSTRACT

We show that a pair of orthogonal partial Latin squares of order n can be embedded in a pair of orthogonal Latin squares of order at most $16n^4$ and all orders greater than or equal to $48n^4$. This paper provides the first direct polynomial order embedding construction for pairs of orthogonal partial Latin squares.

1. Introduction and definitions

Let $[n] = \{0, 1, \ldots, n-1\}$ and N represent a set of n distinct elements. A non-empty subset P of $N \times N \times N$ is said to be a *partial Latin square*, of order n, if for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in P$ and for all distinct $i, j, k \in \{1, 2, 3\}$,

$$x_i = y_i \quad \text{and} \quad x_j = y_j \quad \text{implies} \quad x_k = y_k.$$

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We may think of P as an $n \times n$ array where symbol e occurs in cell (r, c), whenever $(r, c, e) \in P$. We say cell (r, c) is *empty* in P if, for all $e \in N$, $(r, c, e) \notin P$. The *volume* of P is the number of non-empty cells in P. If the volume of P is n^2, then we say that P is a *Latin square*, of order n. We may use a Latin square to define a binary operation; that is, for a Latin square A, define $A(*)$ to be the *quasigroup* where for all $x, y \in N$,

$$x * y = z \quad \text{if and only if} \quad (x, y, z) \in A.$$

Two partial Latin squares P and Q, of the same order are said to be *orthogonal* if they have the same non-empty cells and for all $r_1, c_1, r_2, c_2, x, y \in N$

$$\{(r_1, c_1, x), (r_2, c_2, x)\} \subseteq P \quad \text{implies} \quad \{(r_1, c_1, y), (r_2, c_2, y)\} \nsubseteq Q.$$

Example 1.1. A pair of orthogonal partial Latin squares of order 4.

0	1	2	
2	0	1	3
3		0	
	2		1

0	2	1	
3	1	0	2
1		2	
	0		3

Two partial Latin squares P and Q are said to be *isotopic*, if Q can be obtained by reordering the rows, and/or reordering the columns, and/or relabeling the symbols of P. We say that a partial Latin square P can be *embedded* in a Latin square L if $P \subseteq L$. A pair of orthogonal partial Latin squares (P_1, P_2) is said to be embedded in a pair of orthogonal Latin squares (L_1, L_2) if $P_1 \subseteq L_1$ and $P_2 \subseteq L_2$.

In 1960 Evans [2] proved that a partial Latin square of order n can always be embedded in some Latin square of order t for every $t \geqslant 2n$. In the same paper Evans raised the question as to whether a pair of finite partial Latin squares which are orthogonal can be embedded in a pair of finite orthogonal Latin squares.

It is known (thanks to a series of papers by many authors, see for example [3]) that a pair of orthogonal Latin squares of order n can be embedded in a pair of orthogonal Latin squares of order t if $t \geqslant 3n$, the bound of $3n$ being best possible. Obtaining an analogous result for pairs of orthogonal partial Latin squares has proved to be an extremely challenging problem. Lindner [7] showed that a pair of orthogonal partial Latin squares can always be finitely embedded in a pair of orthogonal Latin squares, however, there was no known method which obtains an embedding of polynomial order (with respect to the order of the partial arrays). In [4], Hilton et al. formulate some necessary conditions for a pair of orthogonal partial Latin squares to be embedded in a pair of orthogonal Latin squares. Jenkins [5] considered a less difficult problem of embedding a single partial Latin square in a Latin square which has an orthogonal mate. His embedding was of order n^2.

The study of orthogonal Latin squares is a very active area of combinatorics (see [1]). It has been shown that a set of $n - 1$ mutually orthogonal Latin squares is equivalent to a projective plane of order n (see [8] for detailed constructions). So the embedding