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# Decompositions of complete uniform hypergraphs into Hamilton Berge cycles ☆

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ABSTRACT

In 1973 Bermond, Germa, Heydemann and Sotteau conjectured that if $n$ divides $\binom{n}{k}$, then the complete $k$-uniform hypergraph on $n$ vertices has a decomposition into Hamilton Berge cycles. Here a Berge cycle consists of an alternating sequence $v_1, e_1, v_2, \dots, v_n, e_n$ of distinct vertices $v_i$ and distinct edges $e_i$ so that each $e_i$ contains $v_i$ and $v_{i+1}$. So the divisibility condition is clearly necessary. In this note, we prove that the conjecture holds whenever $k \geq 4$ and $n \geq 30$. Our argument is based on the Kruskal–Katona theorem. The case when $k = 3$ was already solved by Verrall, building on results of Bermond.



## 1. Introduction

A classical result of Walecki [12] states that the complete graph $K_n$ on $n$ vertices has a Hamilton decomposition if and only if $n$ is odd. (A Hamilton decomposition of a graph $G$ is a set of edge-disjoint Hamilton cycles containing all edges of $G$.) Analogues of

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this result were proved for complete digraphs by Tillson [15] and more recently for (large) tournaments in [10]. Clearly, it is also natural to ask for a hypergraph generalisation of Walecki's theorem.

There are several notions of a hypergraph cycle, the earliest one is due to Berge: A *Berge cycle* consists of an alternating sequence $v_1, e_1, v_2, \dots, v_n, e_n$ of distinct vertices $v_i$ and distinct edges $e_i$ so that each $e_i$ contains $v_i$ and $v_{i+1}$. (Here $v_{n+1} := v_1$ and the edges $e_i$ are also allowed to contain vertices outside $\{v_1, \dots, v_n\}$.) A Berge cycle is a Hamilton (Berge) cycle of a hypergraph $G$ if $\{v_1, \dots, v_n\}$ is the vertex set of $G$ and each $e_i$ is an edge of $G$. So a Hamilton Berge cycle has $n$ edges.

Let $K_n^{(k)}$ denote the complete $k$-uniform hypergraph on $n$ vertices. Clearly, a necessary condition for the existence of a decomposition of $K_n^{(k)}$ into Hamilton Berge cycles is that $n$ divides $\binom{n}{k}$. Bermond, Germa, Heydemann and Sotteau [5] conjectured that this condition is also sufficient. For $k = 3$, this conjecture follows by combining the results of Bermond [4] and Verrall [16].

We show that as long as $n$ is not too small, the conjecture holds for $k \geq 4$ as well.

**Theorem 1.** *Suppose that $4 \leq k < n$, that $n \geq 30$ and that $n$ divides $\binom{n}{k}$. Then the complete $k$-uniform hypergraph $K_n^{(k)}$ on $n$ vertices has a decomposition into Hamilton Berge cycles.*

Recently, Petecki [13] considered a restricted type of decomposition into Hamilton Berge cycles and determined those $n$ for which $K_n^{(k)}$ has such a restricted decomposition.

Walecki's theorem has a natural extension to the case when $n$ is even: in this case, one can show that $K_n - M$ has a Hamilton decomposition, whenever $M$ is a perfect matching. Similarly, the results of Bermond [4] and Verrall [16] together imply that for all $n$, either $K_n^{(3)}$ or $K_n^{(3)} - M$ has a decomposition into Hamilton Berge cycles.

We prove an analogue of this for $k \geq 4$. Note that Theorem 2 immediately implies Theorem 1.

**Theorem 2.** *Let $k, n \in \mathbb{N}$ be such that $3 \leq k < n$.*

(i) *Suppose that $k \geq 5$ and $n \geq 20$ or that $k = 4$ and $n \geq 30$. Let $M$ be any set consisting of less than $n$ edges of $K_n^{(k)}$ such that $n$ divides $|E(K_n^{(k)}) \setminus M|$. Then $K_n^{(k)} - M$ has a decomposition into Hamilton Berge cycles.*
(ii) *Suppose that $k = 3$ and $n \geq 100$. If $\binom{n}{3}$ is not divisible by $n$, let $M$ be any perfect matching in $K_n^{(3)}$, otherwise let $M := \emptyset$. Then $K_n^{(3)} - M$ has a decomposition into Hamilton Berge cycles.*

Note that if $k$ is a prime and $\binom{n}{k}$ is not divisible by $n$, then $k$ divides $n$ and so in this case one can take the set $M$ in (i) to be a union of perfect matchings. Also note that (ii) follows from the results of [4,16]. However, our proof is far simpler, so we also include it in our argument.