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Forbidding just one intersection, for permutations ☆

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ABSTRACT

We prove that for n sufficiently large, if $\mathcal{A}$ is a family of permutations of $\{1,2,\dots,n\}$ with no two permutations in $\mathcal{A}$ agreeing exactly once, then $|\mathcal{A}| \leq (n-2)!$, with equality holding only if $\mathcal{A}$ is a coset of the stabilizer of 2 points. We also obtain a Hilton–Milner type result, namely that if $\mathcal{A}$ is such a family which is not contained within a coset of the stabilizer of 2 points, then it is no larger than the family

$$\begin{aligned}\mathcal{B} = \{\sigma \in S_n :\ &\sigma(1)=1,\ \sigma(2)=2,\\ &\#\{\text{fixed points of } \sigma \geq 5\} \neq 1\}\\ &\cup \{(1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 3\ 2\ 4), (1\ 4\ 2\ 3)\}\end{aligned}$$

We conjecture that for $t \in \mathbb{N}$, and for n sufficiently large depending on t, if $\mathcal{A}$ is family of permutations of $\{1,2,\dots,n\}$ with no two permutations in $\mathcal{A}$ agreeing exactly $t-1$ times, then $|\mathcal{A}| \leq (n-t)!$, with equality holding only if $\mathcal{A}$ is a coset of the stabilizer of t points. This can be seen as a permutation analogue of a conjecture of Erdős on families of k-element sets with a forbidden intersection, proved by Frankl and Füredi in [9].

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1. Introduction

Let X be an n-element set, and let $X^{(k)}$ denote the collection of all k-element subsets of X. We say a family $\mathcal{A} \subset X^{(k)}$ is *t-intersecting* if any two sets in $\mathcal{A}$ share at least t elements, i.e. $|x \cap y| \geq t$ for any $x, y \in \mathcal{A}$. Erdős, Ko and Rado [8] proved in 1961 that if n is sufficiently large depending on k and t, and $\mathcal{A} \subset X^{(k)}$ is t-intersecting, then $|\mathcal{A}| \leq \binom{n-t}{k-t}$, with equality holding only if $\mathcal{A}$ is the family of all k-sets containing some fixed t-element subset of X.

In [7], Erdős asked what happens if we weaken the condition, and just forbid an intersection of size *exactly* $t-1$. Frankl and Füredi [9] proved that for $k \geq 2t$ and for n sufficiently large depending on k, if $\mathcal{A} \subset X^{(k)}$ such that no two sets in $\mathcal{A}$ have intersection of size exactly $t-1$, then $|\mathcal{A}| \leq \binom{n-t}{k-t}$, with equality holding only if $\mathcal{A}$ is the family of all k-sets containing some fixed t-element subset of X.

In this paper, we consider analogues of these problems for the symmetric group S_n, the group of all permutations of $\{1, 2, \dots, n\} =: [n]$. We say that a family of permutations $\mathcal{A} \subset S_n$ is *t-intersecting* if any two permutations in $\mathcal{A}$ agree on at least t points — in other words, for all $\sigma, \tau \in \mathcal{A}$, we have $\#\{i: \ \sigma(i) = \tau(i)\} \geq t$.

Deza and Frankl [2] proved in 1977 that if $\mathcal{A} \subset S_n$ is 1-intersecting, then $|\mathcal{A}| \leq (n-1)!$. The case of equality turned out to be somewhat harder than one might expect; this was resolved in 2003 by Cameron and Ku [1], and independently by Larose and Malvenuto [12], who proved that if $\mathcal{A} \subset S_n$ is an intersecting family of size $(n-1)!$, then $\mathcal{A}$ is a coset of the stabiliser of a point.

Deza and Frankl conjectured in [2] that for any $t \in \mathbb{N}$, if n is sufficiently large depending on t, and $\mathcal{A} \subset S_n$ is t-intersecting, then $|\mathcal{A}| \leq (n-t)!$. This was proved in 2008, by the author and independently by Friedgut and Pilpel, using very similar techniques (specifically, eigenvalue methods, combined with the representation theory of S_n); we have written a joint paper, [6]. We also proved that equality holds only if $\mathcal{A}$ is a *t-coset* of S_n (meaning a coset of the stabiliser of t points), again provided n is sufficiently large depending on t.

Cameron and Ku [1] conjectured that if $\mathcal{A} \subset S_n$ is 1-intersecting, and $\mathcal{A}$ is not contained in any 1-coset, then $\mathcal{A}$ is no larger than the family

$$\{\sigma \in S_n: \ \sigma(1) = 1, \ \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\},$$

which has size $(1 - 1/e + o(1))(n-1)!$. This was proved by the author in [5], using the representation theory of S_n combined with some combinatorial arguments. It can be seen as an analogue of the Hilton–Milner Theorem [10] on 1-intersecting families of r-subsets of $\{1, 2, \dots, n\}$. In [4], the author proved a generalization of the Cameron–Ku conjecture for t-intersecting families, namely that if $\mathcal{A} \subset S_n$ is a t-intersecting family which is not contained within a coset of the stabilizer of t points, then $\mathcal{A}$ is no larger than the family

$$\{\sigma: \ \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t+1\} \cup \{(1\ t+1), \dots, (t\ t+1)\}$$