J. Combin. Theory Ser. A 126 (2014) 177–193

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Counting unlabeled k-trees

Andrew Gainer-Dewar [a], Ira M. Gessel [b,1]

[a] Department of Mathematics, Carleton College, Northfield, MN 55057, United States

[b] Department of Mathematics, Brandeis University, MS 050, Waltham, MA 02453, United States

ARTICLE INFO

Article history:
Received 5 October 2013
Available online 24 May 2014

Keywords:
Unlabeled k-trees
Graphical enumeration
Enumeration under group action

ABSTRACT

We count unlabeled k-trees by properly coloring them in $k+1$ colors and then counting orbits of these colorings under the action of the symmetric group on the colors.

1. Introduction

The class of k-trees may be defined recursively: a k-tree is either a complete graph on k vertices or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique. Thus a 1-tree is an ordinary tree. Fig. 1 shows a 2-tree.

Labeled k-trees may be counted by extensions of the same methods that can be used to count labeled trees. (See, for example, Beineke and Pippert [1], Moon [19], and Foata [8].) However, counting unlabeled k-trees is considerably more difficult.

Harary and Palmer [11] counted unlabeled 2-trees in 1968 (see also Harary and Palmer [12, Section 3.5]) and unlabeled 2-trees were counted in a different way, using the theory

E-mail addresses: againerdewar@carleton.edu (A. Gainer-Dewar), gessel@brandeis.edu (I.M. Gessel).

[1] This work was partially supported by a grant from the Simons Foundation (#229238 to Ira Gessel).

http://dx.doi.org/10.1016/j.jcta.2014.05.002
0097-3165/

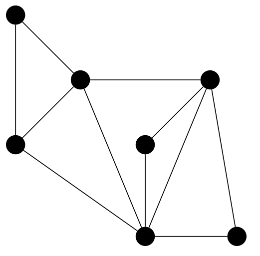

Fig. 1. A 2-tree.

Table 1
The number of k-trees with $n+k$ vertices.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9
1	1	1	1	2	3	6	11	23	47	106
2	1	1	1	2	5	12	39	136	529	2171
3	1	1	1	2	5	15	58	275	1505	9003
4	1	1	1	2	5	15	64	331	2150	15817
5	1	1	1	2	5	15	64	342	2321	18578
"stable"	1	1	1	2	5	15	64	342	2344	19137

of combinatorial species, by Fowler et al. in [9]. Many variations of 2-trees have also been counted [3,7,11,13–15].

The enumeration of unlabeled k-trees for $k > 2$ was a long-standing unsolved problem until the recent solution by Gainer-Dewar [10], also using the theory of combinatorial species. We present here an alternative approach which results in a simpler description of the generating function for unlabeled k-trees; this is given in Theorem 7. The asymptotic growth of the number of k-trees has been analyzed by Drmota and Jin in [6] using our results.

Table 1 gives the number $K_{n,k}$ of k-trees with $n+k$ vertices for small values of n and k; larger tables can be found in [10]. The stability of these numbers for fixed n as k increases and a relation concerning those stable numbers will be explained in Section 3; these "stable k-tree numbers" are shown in the last row of the table. These sequences are given to many more terms as [21, A000055, A054581, A078792, A078793, A201702, A224917] (for $k = 1, 2, 3, 4, 5$ and the stable k-tree numbers respectively).

The usual approach to counting unlabeled trees or tree-like graphs consists of two steps. (See, for example, Bergeron, Labelle, and Leroux [2, Chapters 3 and 4], and Harary and Palmer [12, Chapter 3].)

First, one converts the problem to one of counting certain rootings of these graphs. To do this, Otter [20] introduced the method of "dissimilarity characteristic theorems" in counting unlabeled trees. His theorem relates the numbers of orbits of vertices and edges under the automorphism group of the tree. We will instead use a "dissymmetry theorem" in the style of those introduced by Leroux [16] (see also Leroux and Miloudi [17]); these describe isomorphisms of species and thus may be used to study labeled and unlabeled structures simultaneously. (We will discuss dissimilarity characteristic theorems for trees and 2-trees in Sections 6 and 7.)