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Asymptotic formulas for stacks and unimodal sequences ☆

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ABSTRACT

We study enumeration functions for unimodal sequences of positive integers, where the size of a sequence is the sum of its terms. We survey known results for a number of natural variants of unimodal sequences, including Auluck's generalized Ferrers diagrams, Wright's stacks, and Andrews' convex compositions. These results describe combinatorial properties, generating functions, and asymptotic formulas for the enumeration functions. We also prove several new asymptotic results that fill in the notable missing cases from the literature, including an open problem in statistical mechanics due to Temperley. Furthermore, we explain the combinatorial and asymptotic relationship between partitions, Andrews' Frobenius symbols, and stacks with summits.

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1. Introduction and statement of results

One of the more ubiquitous concepts in enumerative combinatorics is unimodality, as a large number of common combinatorial functions have this property (see Stanley's survey articles [19,20] for many examples and applications). In particular, a *unimodal sequence of size* n is a sequence of positive integers that sum to n such that the terms are monotonically increasing until a peak is reached, after which the terms are monotonically decreasing. This means that the sequence can be written in terms of parts

$$a_1, a_2, \dots, a_r, c, b_s, b_{s-1}, \dots, b_1,$$

that satisfy the inequalities

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq c, \quad \text{and} \tag{1.1}$$

$$c \geq b_s \geq b_{s-1} \geq \cdots \geq b_1 \geq 1. \tag{1.2}$$

When writing examples of unimodal sequences, we typically write the sequences without commas in order to save space. For example, 1244322 is a unimodal sequence of size 18, and all of the unimodal sequences of size 4 are concisely listed as follows:

$$4; \quad 31; \quad 13; \quad 22; \quad 211; \quad 121; \quad 112; \quad 1111.$$

This simple combinatorial definition and its variants have also appeared in the literature under many other names and contexts, including "stacks" [8,24–26], "convex compositions" [3], integer partitions [2,5], and in the study of "saw-toothed" crystal surfaces in statistical mechanics [22]. Furthermore, it is also appropriate to mention the closely related concept of "concave compositions", which were studied in [1–3,6,17], although we will not further discuss these sequences.

The purposes of this paper are twofold. First, we concisely review previous work on enumeration functions related to stacks, unimodal sequences, and compositions, including the corresponding generating functions, and their asymptotic behavior when known. Second, we provide new asymptotic formulas that successfully fill in the notable missing cases from the literature. Along the way we describe some of the analytic techniques that were used in proving the previously known asymptotic formulas, including the theory of modular forms, Euler–MacLaurin summation, and Tauberian theorems; broadly speaking, these methods allow us to use the analytic properties of generating series in order to determine the asymptotic behavior of their coefficients. In order to prove the new asymptotic formulas, we also introduce additional techniques such as the Constant Term Method and Saddle Point Method, which have been less widely used in this combinatorial setting.

Before presenting the definitions of the many variants of unimodal sequences, we first highlight a potential ambiguity in the enumeration of sequences that satisfy (1.1) and (1.2). In particular, if the largest part is not unique, then there are multiple choices