Journal of Combinatorial Theory, Series A 126 (2014) 216–231

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On crown-free families of subsets

Linyuan Lu [1]

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA

ARTICLE INFO

Article history:
Received 12 July 2012
Available online 5 May 2014

Keywords:
Poset
Crown-free families
k-Partite representation
Lubell function
Boolean lattice

ABSTRACT

The crown $\mathcal{O}_{2t}$ is a height-2 poset whose Hasse diagram is a cycle of length $2t$. A family $\mathcal{F}$ of subsets of $[n] := \{1, 2 \ldots, n\}$ is $\mathcal{O}_{2t}$*-free* if $\mathcal{O}_{2t}$ is not a weak subposet of $(\mathcal{F}, \subseteq)$. Let $\mathrm{La}(n, \mathcal{O}_{2t})$ be the largest size of $\mathcal{O}_{2t}$-free families of subsets of $[n]$. De Bonis–Katona–Swanepoel proved $\mathrm{La}(n, \mathcal{O}_4) = \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lceil \frac{n}{2} \rceil}$. Griggs and Lu proved that $\mathrm{La}(n, \mathcal{O}_{2t}) = (1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all even $t \geq 4$. In this paper, we prove $\mathrm{La}(n, \mathcal{O}_{2t}) = (1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all odd $t \geq 7$.

1. Introduction

We are interested in estimating the maximum size of a family of subsets of the n-set $[n] := \{1, \ldots, n\}$ avoiding a given (weak) subposet P. The first of this kind result is Sperner's theorem from 1928 [19], which determined that the maximum size of an antichain in the Boolean lattice $\mathcal{B}_n := (2^{[n]}, \subseteq)$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

For partially ordered sets, (Posets) $P = (P, \leq)$ and $P' = (P', \leq')$, we say P' is a *weak subposet of* P if there exists an injection $f: P' \to P$ that preserves the partial ordering, meaning that whenever $u \leq' v$ in P', we have $f(u) \leq f(v)$ in P (see [20]). Throughout

E-mail address: lu@math.sc.edu.

[1] The author was supported in part by NSF grants DMS 1000475 and DMS 1300547.

http://dx.doi.org/10.1016/j.jcta.2014.04.009

the paper, we mean (weak) subposet. The *height* $h(P)$ of poset P is the maximum size over all chains in P.

We will view a family $\mathcal{F}$ of subsets of $[n]$ as a subposet of $\mathcal{B}_n$. If $\mathcal{F}$ contains no subposet P, we say $\mathcal{F}$ is *P-free*. We are interested in determining the largest size of a P-free family of subsets of $[n]$, denoted $\mathrm{La}(n, P)$.

In this notation, Sperner's theorem [19] gives that $\mathrm{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, where $\mathcal{P}_k$ denotes the path poset on k points, usually called a chain of size k. Let $\mathcal{B}(n, k)$ be the middle k levels in the Boolean lattice $\mathcal{B}_n$ and $\Sigma(n, k) := |\mathcal{B}(n, k)|$. Erdős [10] proved that $\mathrm{La}(n, \mathcal{P}_k) = \sum(n, k-1)$. Let diamonds $\mathcal{D}_k$ be the poset consisting of $A < B_1, \ldots, B_k < C$ and harps $\mathcal{H}(l_1, l_2, \ldots, l_k)$ (assuming $l_1 < l_2 < \cdots < l_k$) be the posets obtained from chains $\mathcal{P}_{l_1}, \mathcal{P}_{l_2}, \ldots, \mathcal{P}_{l_k}$ with their top elements identified and their bottom elements identified. Griggs–Li–Lu [15] showed that the similar results hold for some diamonds $\mathcal{D}_k$ ($k = 3, 4, 7, 8, 9, 15, 16, \ldots$) and all harps $\mathcal{H}(l_1, l_2, \ldots, l_k)$.

For any poset P, we define $e(P)$ to be the maximum m such that for all n, the union of the m middle levels $\mathcal{B}(n, m)$ does not contain P as a subposet. For any $\mathcal{F} \subset 2^{[n]}$, define its Lubell value $h_n(\mathcal{F}) := \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$. Let $\lambda_n(P) = \max\{h_n(\mathcal{F}) \colon \mathcal{F} \subset 2^{[n]}, P\text{-free}\}$. A poset P is called *uniform-L-bounded* if $\lambda_n(P) \le e(P)$ for all n. Griggs–Li [14] proved $\mathrm{La}(n, P) = \Sigma(n, e(P))$ if P is uniform-L-bounded. The uniform-L-bounded posets include $\mathcal{P}_k$ (for any $k \ge 1$), diamonds $\mathcal{D}_k$ (for $k = 3, 4, 7, 8, 9, 15, 16, \ldots$), and harps $\mathcal{H}(l_1, l_2, \ldots, l_k)$ (for $l_1 > l_2 > \cdots > l_k$), and other posets.

For any poset P, Griggs–Lu [16] conjectured the limit $\pi(P) := \lim_{n \to \infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and is an integer. This conjecture is based on various known cases. For $r \ge 2$, let the r-fork $\mathcal{V}_r$ be the poset: $A < B_1, \ldots, B_r$, $r \ge 2$. Katona and Tarján [17] obtained bounds on $\mathrm{La}(n, \mathcal{V}_2)$ that Katona and De Bonis [7] extended in 2007 to general $\mathcal{V}_r$, $r \ge 2$, proving that

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor} \le \mathrm{La}(n, \mathcal{V}_r) \le \left(1 + 2\frac{r-1}{n} + O\left(\frac{1}{n^2}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

While the lower bound is strictly greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, we see that $\mathrm{La}(n, \mathcal{V}_r) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Earlier, Thanh [21] had investigated the more general class of broom-like posets. Griggs and Lu [16] studied the even more general class of baton posets. These are tree posets (meaning that their Hasse diagrams are trees). Griggs and Lu [16] proved that $\pi(T) = 1$ for any tree poset T of height 2. Bukh [4] proved that $\pi(T) = e(T)$ for any general tree poset T.

The most notable unsolved case is the diamond poset D_2. Griggs and Lu first observed $\pi(\mathcal{D}_2) \in [2, 2.296]$. Axenovich, Manske, and Martin [3] came up with a new approach which improved the upper bound to 2.283. Griggs, Li, and Lu [15] further improvess the upper bound to $2.27\dot{3} = 2\frac{3}{11}$. Recently, Kramer–Martin–Young [18] proved $\pi(\mathcal{D}_2) \le 2.25$.

For $k \ge 2$, the crown $\mathcal{O}_{2t}$ is a height-2 poset whose Hasse diagram is a cycle of length $2t$. For $t = 2$, $\mathcal{O}_4$ is also known as the butterfly poset; De Bonis–Katona–Swanepoel [8] proved $\mathrm{La}(n, \mathcal{O}_4) = \Sigma(n, 2)$. Griggs and Lu [16] proved that