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Distances of group tables and latin squares via equilateral triangle dissections

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ABSTRACT

Denote by gdist(p) the least non-zero number of cells that have to be changed to get a latin square from the table of addition modulo p. A conjecture of Drápal, Cavenagh and Wanless states that there exists $c > 0$ such that $\mathrm{gdist}(p) \leqslant c \log(p)$. In this paper the conjecture is proved for $c \approx 7.21$, and as an intermediate result it is shown that an equilateral triangle of side n can be non-trivially dissected into at most $5 \log_2(n)$ integer-sided equilateral triangles. The paper also presents some evidence which suggests that $\mathrm{gdist}(p)/\log(p) \approx 3.56$ for large values of p.

1. Introduction

In this paper we solve a long-standing conjecture about distances of group tables and latin squares. Consider a table of addition modulo prime p; it is a latin square with dimensions $p \times p$. The question is, what is the smallest number gdist(p) of cells we have to change in order to get another latin square?

It has been conjectured [5,2,3] that the answer is $\Theta(\log(p))$, which means there are positive constants c_1, c_2 such that

$$c_1 \log(p) \leqslant \mathrm{gdist}(p) \leqslant c_2 \log(p)$$

for sufficiently large primes p. The lower bound was established before by Drápal and Kepka [6] with the constant $c_1 = e$. An alternative proof of the same estimate was later found by Cavenagh [2], and another proof (with a slightly smaller constant) appears in the paper [4] by Cavenagh and Wanless.

The previously known best upper estimate is due to Drápal [5] and states that $\mathrm{gdist}(p) \leqslant c \log^2(p)$ for some $c > 0$. The method he used relies upon dissections of equilateral triangles into equilateral

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triangles with the special property that no six triangles meet at one point. In this paper we find a construction of such dissections which leads to the logarithmic estimate. To do so we examine dissections of rectangles into squares with the analogous property that no four of them meet at one point.

To be more specific, in Section 2 we show how to dissect a rectangle of size $n \times (n+3)$ into $5\log_4(n) + \frac{3}{2}$ squares and how to adapt the construction to get a dissection of an equilateral triangle of side n into $5\log_2(n)$ triangles. In Section 3 we formulate the conjecture in general for an arbitrary group G and explain the connection to the addition modulo p. The section is concluded by a proof of the conjecture.

Now that we know the asymptotic behavior of $\mathrm{gdist}(p)$ it is natural to ask about the constants in the estimates. The known lower bound with our result for the upper bound give

$$2.72 \approx e < \frac{\mathrm{gdist}(p)}{\log(p)} < 5\log_2(e) \approx 7.21$$

for all primes p. However, there is a conjecture that the limit $\lim_{p\to\infty} \mathrm{gdist}(p)/\log(p)$ exists and is equal to $1/\log(P) \approx 3.56$, where P is such that $P^3 = P + 1$. In Section 4 we present a construction and computational evidence that support such a claim.

2. Dissections of equilateral triangles

In this section we establish a logarithmic upper bound for the number of equilateral triangles needed to dissect an equilateral triangle of side n. Such dissections were originally studied by Tutte [11].

Definition. A *dissection of order k of a rectangle* is a set of k squares of integral side which cover the rectangle and overlap at most on their boundaries. A dissection is $\oplus$*-free* if no four of them share a common point. Let us denote by $r_d(n)$ the minimal order of a $\oplus$-free dissection of a rectangle of size $n \times (n+d)$.

Similarly we define a *dissection of order k of an equilateral triangle* as a set of k equilateral triangles of integral side which cover the triangle and overlap at most on their boundaries. We say that such a dissection is $\circledast$*-free* if no six triangles share a common point, and *non-trivial* if $k > 1$. Let us denote by $t(n)$ the minimal order of a non-trivial $\circledast$-free dissection of an equilateral triangle of side n.

To shorten notation, from now on we write only *triangle* instead of equilateral triangle unless otherwise specified.

The study of dissections of rectangles into squares have been initiated by a joint paper by Brooks, Smith, Stone, and Tutte [1]. In 1965 Trustrum [10] proved that it is possible to find a dissection of an $n \times n$ square into at most $6\log_2(n)$ squares, where $n \geqslant 2$. He did so by examining dissections of rectangles $n \times (n+d)$ with $d \in \{1,2,4\}$. For our purposes we need $\oplus$-free dissections. While it is possible to modify Trustrum's dissections to be such, it turns out to be more efficient to use the case with $d = 3$.

Let us describe an algorithm that dissects a rectangle of size $n \times (n+3)$ for $n \geqslant 2$. Fix the orientation of the rectangle with the shorter side on the left. For convenience, we say that a dissection is *padded* if it has a square of side at least 2 in the upper left corner. Then the algorithm is as follows:

- (A1) For $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$ dissect into $4, 2, 5, 5, 3, 6, 6, 4, 7$ squares respectively such that the dissection is $\oplus$-free and padded. These dissections are completely straightforward as the reader will easily be able to verify.
- (A2) For n of the form $4k + z$ with $k \geqslant 2$, $z \in \{3, 4, 5, 6\}$, depending on z dissect into 3 or 5 squares and a rectangle of size $2k \times 2(k+3)$. Then dissect this rectangle with two times larger tiles recursively. Fig. 1 illustrates the method.