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Generalized frieze pattern determinants and higher angulations of polygons

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A R T I C L E I N F O

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A B S T R A C T

Frieze patterns (in the sense of Conway and Coxeter) are in close connection to triangulations of polygons. Broline, Crowe and Isaacs have assigned a symmetric matrix to each polygon triangulation and computed the determinant. In this paper we consider d-angulations of polygons and generalize the combinatorial algorithm for computing the entries in the associated symmetric matrices; we compute their determinants and the Smith normal forms. It turns out that both are independent of the particular d-angulation, the determinant is a power of $d-1$, and the elementary divisors only take values $d-1$ and 1. We also show that in the generalized frieze patterns obtained in our setting every adjacent 2×2-determinant is 0 or 1, and we give a combinatorial criterion for when they are 1, which in the case $d=3$ gives back the Conway–Coxeter condition on frieze patterns.

1. Introduction

Frieze patterns have been introduced and studied by Conway and Coxeter [7,8]. A frieze pattern (of size n) is an array of n bi-infinite rows of positive integers (arranged as in the example below) such

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that the top and bottom rows consist only of 1's and, most importantly, every set of four adjacent numbers forming a diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the determinant condition $ad - bc = 1$. An example of such a frieze pattern is given by

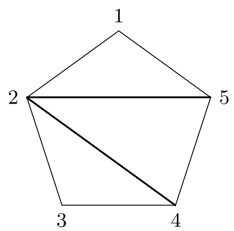
$$\begin{array}{l} \ldots\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\ \ldots \\ \quad \ldots\ 1 \quad 3 \quad 1 \quad 2 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 2\ \ldots \\ \ldots\ 1 \quad 2 \quad 2 \quad 1 \quad 3 \quad 1 \quad 2 \quad 2 \quad 1 \quad 3 \quad 1\ \ldots \\ \quad \ldots\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\ \ldots \end{array}$$

A crucial feature of frieze patterns is that they are invariant under a glide reflection. In the above example, a fundamental domain for the frieze pattern is given by the grey region (green in the web version); the entire pattern is obtained by iteratively performing a glide reflection to this region.

Frieze patterns can be constructed geometrically via triangulations of polygons. For $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a convex n-gon, and consider any triangulation $\mathcal{T}$ of $\mathcal{P}_n$ (necessarily into $n-2$ triangles). We label the vertices of $\mathcal{P}_n$ by $1, \ldots, n$ in counterclockwise order; in the sequel $\mathcal{P}_n$ is always meant to be the convex n-gon together with a fixed labelling.

For each vertex $i \in \{1, \ldots, n\}$ let a_i be the number of triangles of $\mathcal{T}$ incident to the vertex i. Then the sequence $a_1, \ldots, a_n$, repeated infinitely often, gives the second row in a frieze pattern (of size $n-1$).

As an example, consider the case $n = 5$ and the following triangulation of the pentagon



We get for the number of triangles at the vertices the sequence $a_1 = 1$, $a_2 = 3$, $a_3 = 1$, $a_4 = 2$ and $a_5 = 2$, whose repetition gives exactly the second row in the above example of a frieze pattern.

A crucial result of Conway and Coxeter is that every frieze pattern arises in this way from a triangulation.

As a more complicated example, consider the following triangulation of the octagon