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Extending the parking space

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ABSTRACT

The action of the symmetric group S_n on the set Park_n of parking functions of size n has received a great deal of attention in algebraic combinatorics. We prove that the action of S_n on Park_n extends to an action of S_{n+1}. More precisely, we construct a graded S_{n+1}-module V_n such that the restriction of V_n to S_n is isomorphic to Park_n. We describe the S_n-Frobenius characters of the module V_n in all degrees and describe the S_{n+1}-Frobenius characters of V_n in extreme degrees. We give a bivariate generalization $V_n^{(\ell,m)}$ of our module V_n whose representation theory is governed by a bivariate generalization of Dyck paths. A Fuss generalization of our results is a special case of this bivariate generalization.

1. Introduction

This paper is about extending the visible permutation action of S_n on the space Park_n spanned by parking functions of size n to a hidden action of the larger symmetric group S_{n+1}. The S_{n+1}-module we construct will be a subspace of the coordinate ring of the reflection representation of type A_n and will inherit the polynomial grading of this coordinate ring. Using statistics on Dyck paths, Theorem 2 will give an explicit combinatorial formula for the graded S_n-Frobenius character of our module. In Theorem 5 we will describe the extended S_{n+1}-action in extreme degrees.

As far as the authors know, this is the first example of an extension of the S_n-module structure on Park_n to S_{n+1} and the first proof that the S_n-module structure on Park_n extends to S_{n+1}. Our main theorems should be thought of as parallel to several well known extensions, most notably the action of S_{n+1} on the multilinear subspace of the free Lie algebra on $n+1$ symbols, which extends the regular representation of S_n to S_{n+1}. See Section 4 for more on such questions.

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2. Background and main results

A length n sequence $(a_1, \dots, a_n)$ of positive integers is called a *parking function of size* n if its nondecreasing rearrangement $(b_1 \leqslant \cdots \leqslant b_n)$ satisfies $b_i \leqslant i$ for all i.[1] Parking functions were introduced by Konheim and Weiss [5] in the context of computer science, but have seen much application in algebraic combinatorics with connections to Catalan combinatorics, Shi hyperplane arrangements, diagonal coinvariant rings, and rational Cherednik algebras. The set of parking functions of size n is famously counted by $(n+1)^{n-1}$. The $\mathbb{C}$-vector space Park_n spanned by the set of parking functions of size n carries a natural permutation action of the symmetric group S_n on n letters:

$$w.(a_1, \dots, a_n) = (a_{w(1)}, \dots, a_{w(n)}) \tag{2.1}$$

for $w \in S_n$ and $(a_1, \dots, a_n) \in \mathrm{Park}_n$.[2]

A *partition* λ of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$ of nonnegative integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n and define $|\lambda| := n$. We call k the *length* of the partition λ. Observe that we allow zeros as parts of our partitions and that these zeros are included in the length. The *Ferrers diagram* of λ consists of λ_i left justified boxes in the i-th row from the top ('English notation'). If λ is a partition, we define a new partition $\mathrm{mult}(\lambda)$ whose parts are obtained by listing the (positive) part multiplicities in λ in weakly decreasing order. For example, we have that $\mathrm{mult}(4, 4, 3, 3, 3, 1, 0, 0) = (3, 2, 2, 1)$.

We will make use of two orders on partitions in this paper, one partial and one total. The first partial order is *Young's lattice* with relations given by $\lambda \subseteq \mu$ if $\lambda_i \leqslant \mu_i$ for all $i \geqslant 1$ (where we append an infinite string of zeros to the ends of λ and μ so that these inequalities make sense). Equivalently, we have that $\lambda \subseteq \mu$ if and only if the Ferrers diagram of λ fits inside the Ferrers diagram of μ. *Graded reverse lexicographical* (*grevlex*) *order* is the total order on partitions of fixed length n defined by $\lambda \prec \mu$ if either $|\lambda| < |\mu|$ or the final nonzero entry in the vector difference $\lambda - \mu$ is positive. For example, if $n = 6$ we have $(4, 2, 2, 2, 1, 0) \prec (3, 3, 3, 1, 1, 0)$. In particular, either of the relations $\lambda \subseteq \mu$ or $\lambda \preceq \mu$ imply that $|\lambda| \leqslant |\mu|$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we let S_λ denote the *Young subgroup* $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ of S_n. We denote by M^λ the coset representation of S_n given by $M^\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}) \cong_{S_n} \mathbb{C}S_n/S_\lambda$ and we denote by S^λ the irreducible representation of S_n labeled by the partition λ.

Let R_n denote the $\mathbb{C}$-vector space of class functions $S_n \to \mathbb{C}$. Identifying modules with their characters, the set $\{S^\lambda \colon \lambda \vdash n\}$ forms a basis of R_n. The graded vector space $R := \bigoplus_{n \geqslant 0} R_n$ attains the structure of a $\mathbb{C}$-algebra via the induction product $S^\lambda \circ S^\mu := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(S^\lambda \otimes_{\mathbb{C}} S^\mu)$, where $\lambda \vdash n$ and $\mu \vdash m$.

We denote by Λ the ring of symmetric functions (in an infinite set of variables $X_1, X_2, \dots,$ with coefficients in $\mathbb{C}$). The $\mathbb{C}$-algebra Λ is graded and we denote by Λ_n the homogeneous piece of degree n. Given a partition λ, we denote the corresponding Schur function by s_λ and the corresponding complete homogeneous symmetric function by h_λ.

The *Frobenius character* is the graded $\mathbb{C}$-algebra isomorphism $\mathrm{Frob} : R \to \Lambda$ induced by setting $\mathrm{Frob}(S^\lambda) = s_\lambda$. It is well known that we have $\mathrm{Frob}(M^\lambda) = h_\lambda$. Generalizing slightly, if $V = \bigoplus_{k \geqslant 0} V(k)$ is a graded S_n-module, define the *graded Frobenius character* $\mathrm{grFrob}(V; q) \in \Lambda \otimes_{\mathbb{C}} \mathbb{C}[[q]]$ to be the formal power series in q with coefficients in Λ given by $\mathrm{grFrob}(V; q) := \sum_{k \geqslant 0} \mathrm{Frob}(V(k))q^k$.

A *Dyck path of size* n is a lattice path D in $\mathbb{Z}^2$ consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(0, 0)$, ends at (n, n), and stays weakly above the line $y = x$. A maximal contiguous sequence of vertical steps in D is called a *vertical run* of D.

[1] The terminology arises from the following situation. Consider a linear parking lot with n parking spaces and n cars that want to park in the lot. For $1 \leqslant i \leqslant n$, car i wants to park in the space a_i. At stage i of the parking process, car i parks in the first available spot $\geqslant a_i$, if any such spots are available. If no such spots are available, car i leaves the lot. The driver preference sequence $(a_1, \dots, a_n)$ is a parking function if and only if all cars are able to park in the lot.

[2] Here we adopt the symmetric group multiplication convention that says, for example, $(1, 2)(2, 3) = (1, 2, 3)$ so that this is a *left* action.