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# Cayley compositions, partitions, polytopes, and geometric bijections

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A B S T R A C T

In 1857, Cayley showed that certain sequences, now called *Cayley compositions*, are equinumerous with certain partitions into powers of 2. In this paper we give a simple bijective proof of this result and a geometric generalization to equality of Ehrhart polynomials between two convex polytopes. We then apply our results to give a new proof of *Braun's conjecture* proved recently by the authors [15].


## 0. Introduction and main results

Partition theory is a classical field with a number of advanced modern results and applications. Its long and tumultuous history left behind a number of beautiful results which are occasionally brought to light to wide acclaim. The story of the so called *Cayley compositions* is a prime example of this. Introduced and studied by Cayley in 1857 [7], they were rediscovered by Minc [17], and remained largely forgotten until Andrews, Paule, Riese and Strehl [2] resurrected and christened them in 2001. This is when things became really interesting.

**Theorem 1.** *(See Cayley [7].) The number of integer sequences* $(a_1, \ldots, a_n)$ *such that* $1 \leqslant a_1 \leqslant 2$*, and* $1 \leqslant a_{i+1} \leqslant 2a_i$ *for* $1 \leqslant i < n$*, is equal to the total number of partitions of integers* $N \in \{0, 1, \ldots, 2^n - 1\}$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

Our first result is a long elusive bijective proof of Cayley's theorem, and its several extensions. Our bijection construction is geometric, based on our approach in [18].

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Denote by $\mathcal{A}_n$ the set of sequences $(a_1, \ldots, a_n)$ satisfying the conditions of the theorem, which are called *Cayley compositions*. Denote by $\mathcal{B}_n$ the set of partitions into powers of 2 as in the theorem, which we call *Cayley partitions*. Now Theorem 1 states that $|\mathcal{A}_n| = |\mathcal{B}_n|$. For example,

$$\mathcal{A}_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(2,4)\}, \qquad \mathcal{B}_2 = \{21, 2, 1^3, 1^2, 1, \varnothing\},$$

so $|\mathcal{A}_2| = |\mathcal{B}_2| = 6$. Following [4], define the *Cayley polytope* $\boldsymbol{A}_n$ to be the convex hull of all Cayley compositions $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

The main result of this paper is the following geometric extension of Theorem 1. Recall that the *Ehrhart polynomial* $\mathcal{E}_P(t)$ of a lattice polytope $P \subset \mathbb{R}^n$ is defined by

$$\mathcal{E}_P(k) = \#\{kP \cap \mathbb{Z}^n\},$$

where $kP$ denotes the $k$-fold dilation of $P$, $k \in \mathbb{N}$ (see e.g. [3]).

**Theorem 2.** *Let $\mathcal{B}_n$ be the set of Cayley partitions, where a partition of the form $(2^{n-1})^{m_1}(2^{n-2})^{m_2}\ldots 1^{m_n}$ is identified with an integer point $(m_1, m_2, \ldots, m_n) \in \mathbb{R}^n$. Now let $\boldsymbol{B}_n = \operatorname{conv}\mathcal{B}_n$. Then $\mathcal{E}_{\boldsymbol{A}_n}(t) = \mathcal{E}_{\boldsymbol{B}_n}(t)$.*

In particular, when $t = 1$, we obtain Cayley's theorem. Our proof is based on an explicit volume-preserving map $\varphi : \boldsymbol{B}_n \to \boldsymbol{A}_n$, which satisfies a number of interesting properties. In particular, when restricted to integer points, this map gives the bijection $\varphi : \mathcal{B}_n \to \mathcal{A}_n$ mentioned above (see Proposition 6).

In [4], Ben Braun made an interesting conjecture about the volume of $\boldsymbol{A}_n$, which was recently proved by the authors [15]. Denote by $\mathcal{C}_n$ the set of connected graphs on $n$ nodes, and let $C_n = |\mathcal{C}_n|$.

**Theorem 3** *(Formerly Braun's conjecture). (See [15].) Let $\mathcal{A}_n \subset \mathbb{R}^n$ be the set of Cayley compositions, and let $\boldsymbol{A}_n = \operatorname{conv}\mathcal{A}_n$ be the* Cayley polytope. *Then* $\operatorname{vol}\boldsymbol{A}_n = C_{n+1}/n!$.

Combined with Theorem 2, we immediately have $\operatorname{vol}\boldsymbol{B}_n = \operatorname{vol}\boldsymbol{A}_n$, and conclude:

**Corollary 4.** *Let $\mathcal{B}_n$ be the polytope defined above. Then* $\operatorname{vol}\boldsymbol{B}_n = C_{n+1}/n!$.

Curiously, one can also use $\operatorname{vol}\boldsymbol{B}_n = \operatorname{vol}\boldsymbol{A}_n$ in reverse, and derive Theorem 3 from Theorem 2 and known results on *Stanley–Pitman polytopes* (see below).

The rest of this paper is structured as follows. In Section 1 we prove Theorems 1 and 2 using an explicit bijection $\varphi$. Some applications are given in Section 2, followed by a new proof of Theorem 3 in Section 3. We finish with final remarks in Section 4.

## 1. Bijection construction

Recall from [4,15] (or observe directly from the definition) that Cayley polytope $\boldsymbol{A}_n \subset \mathbb{R}^n$ is defined by the following inequalities:

$$1 \leqslant x_1 \leqslant 2, \qquad 1 \leqslant x_2 \leqslant 2x_1, \qquad \ldots, \qquad 1 \leqslant x_n \leqslant 2x_{n-1}.$$

Consider a basis

$$\boldsymbol{e}_1 = (1, 2, 4, \ldots, 2^{n-1}), \qquad \boldsymbol{e}_2 = (0, 1, 2, \ldots, 2^{n-2}), \qquad \ldots, \qquad \boldsymbol{e}_n = (0, 0, \ldots, 1),$$

and a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$\varphi(b_1, b_2, \ldots, b_n) = (2, 4, \ldots, 2^n) - \sum_{i=1}^{n} b_i \boldsymbol{e}_i.$$