Journal of Combinatorial Theory, Series A 123 (2014) 104–122

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Counting trees using symmetries

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ARTICLE INFO

Article history:
Received 4 June 2012
Available online 20 December 2013

Keywords:
Cayley trees
Vertical profile of trees
Lagrange inversion
Plane trees
Plane cacti

ABSTRACT

We prove a new formula for the generating function of multitype Cayley trees counted according to their degree distribution. Using this formula we recover and extend several enumerative results about trees. In particular, we extend some results by Knuth and by Bousquet-Mélou and Chapuy about embedded trees. We also give a new proof of the multivariate Lagrange inversion formula. Our strategy for counting trees is to exploit symmetries of refined enumeration formulas: proving these symmetries is easy, and once the symmetries are proved the formulas follow effortlessly. We also adapt this strategy to recover an enumeration formula of Goulden and Jackson for cacti counted according to their degree distribution.

1. Introduction

The enumeration of trees is a very classical subject. For instance, there is a well-known formula for the number of *unitype Cayley trees*. Recall that a unitype Cayley tree with n vertices is a connected acyclic graph with vertex set $[n]=\{1,\dots,n\}$. There are n^{n-2} such trees, and there is a very simple formula for the generating function of Cayley trees counted according to their degree distribution. Namely,

$$\sum_{\substack{T\,\text{Cayley tree}\\ \text{with vertex set }[n]}} x_1^{\deg(1)}x_2^{\deg(2)}\cdots x_n^{\deg(n)} = x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)^{n-2}, \tag{1}$$

where $\deg(i)$ is the degree of vertex i.

In this paper we consider *multitype Cayley trees*, that is, trees in which vertices have both a *type* and a *label*. We obtain a formula extending (1) from the unitype setting to the multitype setting

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[1] O.B. acknowledges support from NSF grant DMS-1308441, and ERC Explore-Maps.

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2013.12.001

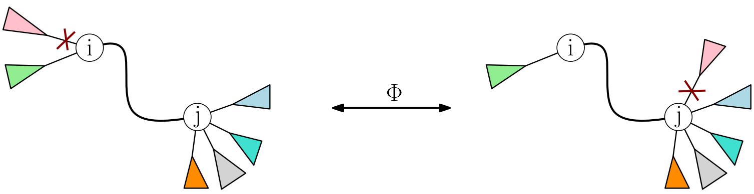

Fig. 1. The bijection Φ between the sets $\mathcal{T}_{\gamma}^{i,j}$ and $\mathcal{T}_{\gamma'}^{j,i}$.

(Theorem 2). More precisely, our formula gives the generating function of rooted multitype Cayley trees counted according to the number of children of each type of each vertex. Our formula is surprisingly simple, and from it we derive many enumerative corollaries in Section 3. In particular, we recover and extend the results of Knuth [13], and the recent results of Bousquet-Mélou and Chapuy [4] about "embedded trees". We also obtain a short proof of the multivariate Lagrange inversion formula [7] in Section 4. Our strategy for counting trees is to exploit symmetries of refined enumeration formulas, and we also use this strategy in order to recover a formula of Goulden and Jackson for counting cacti according to their degree distribution in Section 5. We mention lastly that because we count trees according to their vertex degrees, our results could equivalently be stated in terms of plane trees instead of Cayley trees (see Section 5 for a more detailed discussion). Also, our results can easily be extended in order to count rooted forests (see Corollary 3).

In order to illustrate our approach for counting trees, we give a new proof of (1). There are already many beautiful proofs of this formula including Prüfer's code bijection [17], Joyal's endofunction approach [12], Pitman's double counting argument [16], the matrix-tree theorem [15, Chapter 5], and recursive approaches [18, Chapter 5.3]. Our method is different: we start by proving the "symmetries" in formula (1) and use them at our advantage in order to enumerate Cayley trees.

First observe that a Cayley tree with n vertices has $n-1$ edges, hence the degrees of its vertices sum to $2n-2$. Given a tuple of positive integers $\gamma = (\gamma_1, \dots, \gamma_n)$ summing to $2n-2$, we denote by $\mathcal{T}_\gamma$ the set of Cayley trees with n vertices such that vertex i has degree γ_i for all $i \in [n]$. We first claim that the cardinalities of the sets $\mathcal{T}_\gamma$ are related to one another by simple factors:

Lemma 1. *Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be tuple of positive integers summing to $2n-2$. Let $i, j \in [n]$ and let $\gamma' = (\gamma'_1, \dots, \gamma'_n)$ be defined by $\gamma'_i = \gamma_i - 1$, $\gamma'_j = \gamma_j + 1$ and $\gamma'_k = \gamma_k$ for $k \neq i, j$. Then*

$$(\gamma_i - 1)|\mathcal{T}_\gamma| = (\gamma'_j - 1)|\mathcal{T}_{\gamma'}|.$$

Proof. The proof is illustrated in Fig. 1. Let $\mathcal{T}_\gamma^{i,j}$ be the set of trees in $\mathcal{T}_\gamma$ with a marked edge incident to vertex i not in the path between vertices i and j. Clearly $|\mathcal{T}_\gamma^{i,j}| = (\gamma_i - 1)|\mathcal{T}_\gamma|$. Moreover, there is an obvious bijection Φ between $\mathcal{T}_\gamma^{i,j}$ and $\mathcal{T}_{\gamma'}^{j,i}$: given a marked tree $T \in \mathcal{T}_\gamma^{i,j}$, the tree $\Phi(T) \in \mathcal{T}_{\gamma'}^{j,i}$ is obtained by ungluing the marked edge from vertex i, and gluing it to vertex j. $\square$

Using Lemma 1 repeatedly, we can express $|\mathcal{T}_\gamma|$ in terms of $|\mathcal{T}_\kappa|$, where $\kappa = (n-1, 1, 1, \dots, 1)$. Indeed,

$$\begin{aligned} |\mathcal{T}_\gamma| &= \frac{\gamma_1(\gamma_1+1)\cdots(\gamma_1+\gamma_2-2)}{(\gamma_2-1)!}|\mathcal{T}_{\gamma_1+\gamma_2-1,1,\gamma_3,\dots,\gamma_n}| \\ &= \frac{\gamma_1(\gamma_1+1)\cdots(\gamma_1+\gamma_2+\cdots+\gamma_n-n)}{(\gamma_2-1)!\,(\gamma_3-1)!\,\dots\,(\gamma_n-1)!}|\mathcal{T}_\kappa| \\ &= \binom{n-2}{\gamma_1-1, \gamma_2-1, \dots, \gamma_n-1}|\mathcal{T}_\kappa|. \end{aligned} \tag{2}$$