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On balanced incomplete block designs with specified weak chromatic number

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ABSTRACT

A weak c-colouring of a balanced incomplete block design (BIBD) is a colouring of the points of the design with c colours in such a way that no block of the design has all of its vertices receive the same colour. A BIBD is said to be weakly c-chromatic if c is the smallest number of colours with which the design can be weakly coloured. In this paper we show that for all $c \geqslant 2$ and $k \geqslant 3$ with $(c, k) \neq (2, 3)$, the obvious necessary conditions for the existence of a (v, k, λ)-BIBD are asymptotically sufficient for the existence of a weakly c-chromatic (v, k, λ)-BIBD.

1. Introduction

A *balanced incomplete block design* of order v, block size k and index λ, denoted a (v, k, λ)-BIBD, is a pair $(V, \mathcal{B})$ such that V is a set of v elements (called points) and $\mathcal{B}$ is a collection of k element subsets of V (called blocks) such that each unordered pair of points in V is contained in exactly λ blocks in $\mathcal{B}$. A *partial* (v, k, λ)-BIBD is defined similarly except that each pair of points in V must be contained in at most λ blocks in $\mathcal{B}$.

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For a positive integer c, a weak c-colouring of a (partial) (v,k,λ)-BIBD is a colouring of the points of the design with c colours in such a way that no block of the design has all of its points receive the same colour. A (partial) (v,k,λ)-BIBD is said to be *weakly c-chromatic*, or to have *weak chromatic number* c, if c is the smallest number of colours with which the design can be weakly coloured. Since weak colourings are the only colourings of designs we will consider in this paper, we will often omit the adjectives 'weak' and 'weakly' in what follows.

It is obvious that if there exists a (v,k,λ)-BIBD then

(i) $\lambda(v-1) \equiv 0 \pmod{k-1}$; and
(ii) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

Wilson [20] famously proved that (i) and (ii) are asymptotically sufficient for the existence of a (v,k,λ)-BIBD. That is, for any positive integers k and λ with $k \geqslant 3$, there exists an integer $N'(k,\lambda)$ such that if $v \geqslant N'(k,\lambda)$ then (i) and (ii) are sufficient for the existence of a (v,k,λ)-BIBD. In this paper, we will extend Wilson's result to c-chromatic BIBDs by showing that, for any positive integers c, k and λ, such that $c \geqslant 2$, $k \geqslant 3$ and $(k,c) \neq (3,2)$, (i) and (ii) are asymptotically sufficient for the existence of a c-chromatic (v,k,λ)-BIBD. For the sake of brevity, we will call positive integers v which satisfy (i) and (ii) (k,λ)-*admissible*. Note that if an integer v is (k,λ)-admissible then so is every positive integer congruent to v modulo $k(k-1)$.

Weak colourings were first introduced in the context of hypergraphs, and this naturally led to the study of weak colourings of block designs. A simple counting argument [16] shows that 2-chromatic $(v,3,\lambda)$-BIBDs exist only for $v \leqslant 4$. For a positive integer λ it is known that a 2-chromatic $(v,4,\lambda)$-BIBD exists for each $(4,\lambda)$-admissible integer v, with almost all of the problem solved in [9] and [10] and the outstanding cases resolved in [17] and [6]. Ling [14] has proved that a 2-chromatic $(v,5,1)$-BIBD exists for each $(5,1)$-admissible integer v. It has been shown by de Brandes, Phelps and Rödl [4] that for all integers $c \geqslant 3$ there is an integer $N(c,3,1)$ such that for all $(3,1)$-admissible integers $v \geqslant N(c,3,1)$ there is a c-chromatic $(v,3,1)$-BIBD. The analogous result for $(v,4,1)$-BIBDs has been proved by Linek and Wantland [13]. For a survey of colourings of block designs see [17]. The main result of this paper is as follows.

Theorem 1.1. *Let c, k and λ be positive integers such that $c \geqslant 2$, $k \geqslant 3$ and $(c,k) \neq (2,3)$. Then there is an integer $N(c,k,\lambda)$ such that there exists a weakly c-chromatic (v,k,λ)-BIBD for all (k,λ)-admissible integers $v \geqslant N(c,k,\lambda)$.*

In Section 2 we give some definitions that we will require throughout the paper and prove a number of preliminary results. Sections 3, 4 and 5 deal with BIBDs with block size at least 4. In Section 3 we find various examples of 2-chromatic BIBDs, and these are then used in Section 4 to obtain various examples of c-chromatic BIBDs for each $c \geqslant 2$. In Section 5 we are then able to use results from Sections 2, 3 and 4 to demonstrate the