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A three shuffle case of the compositional parking function conjecture

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ABSTRACT

We prove here that the polynomial $\langle \nabla \mathbf{C}_p 1, e_a h_b h_c \rangle$ q,t-enumerates, by the statistics *dinv* and *area*, the parking functions whose supporting Dyck path touches the main diagonal according to the composition $p \models a+b+c$ and has a reading word which is a shuffle of one decreasing word and two increasing words of respective sizes a, b, c. Here $\mathbf{C}_p 1$ is a rescaled Hall–Littlewood polynomial and ∇ is the Macdonald eigen-operator introduced by Bergeron and Garsia. This is our latest progress in a continued effort to settle the decade old *shuffle conjecture* of Haglund et al. This result includes as special cases all previous results connected with the shuffle conjecture such as the q,t-Catalan, Schröder and two shuffle results of Haglund as well as their compositional refinements recently obtained by the authors. It also confirms the possibility that the approach adopted by the authors has the potential to yield a resolution of the shuffle parking function conjecture as well as its compositional refinement more recently proposed by Haglund, Morse and Zabrocki.

1. Introduction

A parking function may be visualized as a Dyck path in an $n \times n$ lattice square with the cells adjacent to the vertical edges of the path labeled with a permutation of the integers $\{1, 2, \ldots, n\}$ in a column increasing way (see attached figure). We will borrow from parking function language by calling these labels *cars*. The corresponding preference function is simply obtained by specifying that *car i* prefers to park at the bottom of its column. This visual representation, which has its origins in [4], uses the Dyck path to assure that the resulting preference function parks the cars.

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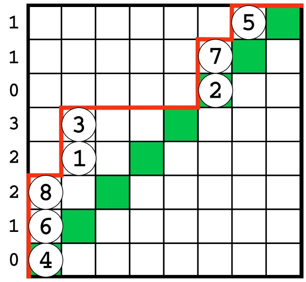

Fig. 1. A typical example of a parking function.

The sequence of cells that joins the SW corner of the lattice square to the NE corner will be called the *main diagonal* or the 0-diagonal of the parking function. The successive diagonals above the main diagonal will be referred to as diagonals $1, 2, \ldots, n-1$ respectively. On the left of the adjacent display we have listed the diagonal numbers of the corresponding cars. It is also convenient to represent a parking function as a two line array

$$PF = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \tag{1.1}$$

where $v_1, v_2, \ldots, v_n$ are the cars as we read them by rows from bottom to top and $u_1, u_2, \ldots, u_n$ are their corresponding diagonal numbers. From the geometry of the above display we can immediately see that for a two line array to represent a parking function it is necessary and sufficient that we have

$$u_1 = 0 \quad \text{and} \quad 0 \leqslant u_i \leqslant u_{i-1} + 1 \tag{1.2}$$

with $V = (v_1, v_2, \ldots, v_n)$ a permutation satisfying the condition

$$u_i = u_{i-1} + 1 \quad \Longrightarrow \quad v_i > v_{i-1}. \tag{1.3}$$

For instance the parking function in Fig. 1 corresponds to the following two line array

$$PF = \begin{bmatrix} 4 & 6 & 8 & 1 & 3 & 2 & 7 & 5 \\ 0 & 1 & 2 & 2 & 3 & 0 & 1 & 1 \end{bmatrix}.$$

Parking functions will be enumerated here by means of a weight that is easily defined in terms of their two line arrays. To this end, let us denote by $\sigma(PF)$ the permutation obtained by successive right to left readings of the components of $V = (v_1, v_2, \ldots, v_n)$ according to decreasing values of $u_1, u_2, \ldots, u_n$. We will call $\sigma(PF)$ the *diagonal word* of PF. We will also let $\text{ides}(PF)$ denote the descent set of the inverse of $\sigma(PF)$.

This given, each parking function is assigned the weight

$$w(PF) = t^{\text{area}(PF)} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)}[X]$$