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Hypergraph Turán numbers of linear cycles

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A B S T R A C T

A *k-uniform linear cycle* of length ℓ, denoted by $\mathbb{C}_\ell^{(k)}$, is a cyclic list of k-sets $A_1, \dots, A_\ell$ such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint. For all $k \geqslant 5$ and $\ell \geqslant 3$ and sufficiently large n we determine the largest size of a k-uniform set family on $[n]$ not containing a linear cycle of length ℓ. For odd $\ell = 2t+1$ the unique extremal family $\mathcal{F}_S$ consists of all k-sets in $[n]$ intersecting a fixed t-set S in $[n]$. For even $\ell = 2t+2$, the unique extremal family consists of $\mathcal{F}_S$ plus all the k-sets outside S containing some fixed two elements. For $k \geqslant 4$ and large n we also establish an exact result for so-called *minimal cycles*. For all $k \geqslant 4$ our results substantially extend Erdős's result on largest k-uniform families without $t+1$ pairwise disjoint members and confirm, in a stronger form, a conjecture of Mubayi and Verstraëte. Our main method is the delta system method.

1. Introduction

The delta system method is a very useful tool for set system problems. It was fully developed in a series of papers including [11] and [8]. It was successfully used for starlike configurations in [8] and [14] (see also [19] for a related result) and recently also for

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larger configurations (as paths and trees) in [12] and [15]. In this paper we apply the delta system method, particularly tools from [11] and [8], to determine, for all $k \geqslant 5$ and large n, the Turán numbers of certain hypergraphs called *k-uniform linear cycles*. This confirms, in a stronger form, a conjecture of Mubayi and Verstraëte [24] for $k \geqslant 5$ and adds to the limited list of hypergraphs whose Turán numbers have been known either exactly or asymptotically.

We organize the paper as follows. Sections 2 and 3 contain definitions concerning hypergraphs. Section 4 gives a rough upper bound establishing the correct order of the magnitude. Section 6 contains the statements of the main results. Section 7 introduces the delta system method and lemmas needed for the linear cycle problem and Sections 8–10 contain proofs.

2. Definitions: shadows, degrees, delta systems

A *hypergraph* $\mathcal{F} = (V, \mathcal{E})$ consists of a set V of vertices and a set $\mathcal{E}$ of edges, where each edge is a subset of V. If V has n vertices, then it is often convenient to just assume that $V = [n] = \{1, 2, \dots, n\}$. Let $\binom{V}{k}$ denote the collection of all the k-subsets of V. If all the edges of $\mathcal{F}$ are k-subsets of V, then we write $\mathcal{F} \subseteq \binom{V}{k}$ and say that $\mathcal{F}$ is a *k-uniform* hypergraph, or a *k-graph* for brevity, on V. Note that the usual graphs are precisely 2-graphs on respective vertex sets. A hypergraph $\mathcal{F} = (V, \mathcal{E})$ is also often times called a *set system* or *set family* on V with its edges referred to as the *members* of the set system/family. A k-graph $\mathcal{F}$ is *k-partite* if its vertex set V can be partitioned into k subsets $V_1, \dots, V_k$ such that each edge of $\mathcal{F}$ contains precisely one vertex from each V_i.

The *shadow* of $\mathcal{F}$, denoted by $\partial(\mathcal{F})$, is defined as

$$\partial(\mathcal{F}) = \{D\colon\ \exists F \in \mathcal{F},\ D \subsetneq F\}.$$

Here, we treat $\emptyset$ as a member of $\partial(\mathcal{F})$. We define the *p-shadow* of $\mathcal{F}$ to be

$$\partial_p(\mathcal{F}) = \bigl\{D\colon\ D \in \partial(\mathcal{F}),\ |D| = p\bigr\}.$$

The Lovász [21] version of the Kruskal–Katona theorem states that if $\mathcal{F}$ is a k-graph of size $|\mathcal{F}| = \binom{x}{k}$ where $x \geqslant k-1$ is a real number, then for $k \geqslant p \geqslant 1$

$$\bigl|\partial_p(\mathcal{F})\bigr| \geqslant \binom{x}{p}. \tag{2.1}$$

Let $\mathcal{F}$ be a hypergraph on $[n]$ and $D \subseteq V(\mathcal{F})$. The *degree* $\deg_{\mathcal{F}}(D)$ of D in $\mathcal{F}$, is defined as

$$\deg_{\mathcal{F}}(D) = \bigl|\{F\colon\ F \in \mathcal{F},\ D \subseteq F\}\bigr|.$$

A family of sets $F_1, \dots, F_s$ is said to form an *s-star* or *Δ-system* of size s with *kernel* D if $F_i \cap F_j = D$ for all $1 \leqslant i < j \leqslant s$ and $\forall i \in [s]$, $F_i \setminus D \neq \emptyset$. The sets $F_1, \dots, F_s$ are