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Two notions of unit distance graphs

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ABSTRACT

A *faithful (unit) distance graph* in $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of the d-dimensional Euclidean space, where two vertices are adjacent if and only if the Euclidean distance between them is exactly 1. A *(unit) distance graph* in $\mathbb{R}^d$ is any subgraph of such a graph.

In the first part of the paper we focus on the differences between these two classes of graphs. In particular, we show that for any fixed d the number of faithful distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1+o(1))dn\log_2 n}$, and give a short proof of the known fact that the number of distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1-1/\lfloor d/2\rfloor+o(1))n^2/2}$. We also study the behavior of several Ramsey-type quantities involving these graphs.

In the second part of the paper we discuss the problem of determining the minimum possible number of edges of a graph which is not isomorphic to a faithful distance graph in $\mathbb{R}^d$.

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1. Introduction

1.1. Background

We study the differences between the following two well-known notions of (unit) distance graphs:

Definition 1. A graph $G = (V, E)$ is a *(unit) distance graph in* $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y)\colon\ x, y \in V,\ |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.

Definition 2. A graph $G = (V, E)$ is a *faithful (unit) distance graph in* $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and $E = \{(x, y)\colon\ x, y \in V,\ |x - y| = 1\}$.

We say that a graph G is realized as a (faithful) distance graph in $\mathbb{R}^d$, if it is isomorphic to some (faithful) distance graph in $\mathbb{R}^d$. Denote by $\mathcal{D}(d)$ ($\mathcal{D}_n(d)$) the set of all labelled distance graphs in $\mathbb{R}^d$ (of order n). Similarly, denote by $\mathcal{FD}(d)$ ($\mathcal{FD}_n(d)$) the set of all labelled faithful distance graphs in $\mathbb{R}^d$ (of order n).

Distance graphs appear in the investigation of two well-studied problems. The first is the problem of determining the chromatic number $\chi(\mathbb{R}^d)$ of the d-dimensional space:

$$\chi\big(\mathbb{R}^d\big) = \min\big\{m \in \mathbb{N}\colon\ \mathbb{R}^d = H_1 \cup \cdots \cup H_m\colon\ \forall i,\ \forall x, y \in H_i,\ |x - y| \neq 1\big\}.$$

The second is the investigation of the maximum possible number $f_2(n)$ of pairs of points at unit distance apart in a set of n points in the plane $\mathbb{R}^2$. Distance graphs arise naturally in the context of both problems. Indeed,

$$\chi\big(\mathbb{R}^d\big) = \max_{G \in \mathcal{D}(d)} \chi(G) = \max_{G \in \mathcal{FD}(d)} \chi(G),$$
$$f_2(n) = \max_{G \in \mathcal{D}_n(2)} \big|E(G)\big| = \max_{G \in \mathcal{FD}_n(2)} \big|E(G)\big|.$$

Thus, in the study of these two extremal problems it does not matter whether we consider distance graphs or faithful distance graphs. However, there is a substantial difference between the sets $\mathcal{D}(d)$ and $\mathcal{FD}(d)$. This difference is discussed in the theorems that appear in what follows.

1.2. The main results

The first theorem provides some classes of graphs that are (or are not) distance or faithful distance graphs in $\mathbb{R}^d$. A surprising aspect of Theorem 1.1 is that for any d there are bipartite graphs that are not faithful distance graphs in $\mathbb{R}^d$.